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ABSTRACT

This collection of materials includes six units dealing with applications of matrix methods. These are: 105-Food Service Management; 107-Markov Chains; 108-Electrical Circuits; 109-Food Service and Dietary Requirements; 111-Fixed Point and Absorbing Markov Chains; and 112-Analysis of Linear Circuits. The units contain exercises and model exams, with answers to at least some exercises and to all test questions. This document set also contains four sections on derivatives of trigonometric functions: 158-Challenge Problems; 159-Formulating Conjectures About the Derivatives; 160-Verifying Conjectures About the Derivatives; and model exams and answers to these test problems. The final module included is 162-Determining Constants of Integration. Exercises and problem solutions are included in this unit. (MP)

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APPLICATIONS

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380 Green Street
Cambridge, MA 02139

MODULE 105 & 109

Food Service Management *(U105)* and Applications of Matrix Methods: Food Service and Dietary Requirements *(U109)*

by Sister Mary K. Keller

$$\begin{array}{l|cccc}
 & \text{pound cake} & \text{scrambled eggs} & \text{omelet} & \text{beef strog} \\
\hline
\text{eggs} & 1 & .5 & 1 & 0 \\
\text{flour} & 1 & 0 & .25 & 0 \\
\text{sugar} & 1 & 0 & 0 & 0 \\
\text{butter} & 1 & .25 & .25 & 0 \\
\text{beef stroganoff} & 0 & 0 & 0 & 1
\end{array}$$

Applications of Matrix Methods

SE 036 477

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Intermodular Description Sheet: UMAP Units 105 and 109

Title: FOOD SERVICE MANAGEMENT (U105) and APPLICATIONS OF MATRIX METHODS TO FOOD SERVICE AND DIETARY REQUIREMENTS (U109)

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Classification: APPL MATRIX/FOOD SERVICE MGMT

Suggested Support Material: Because of the length of the nutrient file it is not reproduced in this unit. The following reference may be obtained from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402:

Nutritive Value of Foods
Home and Garden Bulletin No. 72
United States Department of Agriculture

Prerequisite Skills:

1. Understand the concept of a matrix.
2. Be able to perform matrix multiplication.
3. Understand the inverse of a matrix.

Output Skills:

1. Interpret the meaning of the entries in a matrix which was formulated from a specific problem.
2. Interpret the results of matrix multiplication for a specific problem.
3. Be able to use an existing computer program to perform matrix multiplication; i.e., prepare correct input for the program.
4. Given a suitable problem, formulate matrices to be used in its solution.
5. Solve a problem which has been formulated in terms of matrices, using matrix multiplication if appropriate.
6. Be able to interpret the results of matrix operations on recipe and nutrient matrices.
7. Recognize the operations required to determine nutrient content of recipes and costs.
8. Construct recipe and nutrient matrices.

Other Related Units:

Computer Graphics *(U106)* and Applications of Matrix Methods: Three Dimensional Computer Graphics *(U110)*
Markov Chains *(U107)* and Applications of Matrix Methods: Fixed-Point and Absorbing Markov Chains *(U111)*
Electrical Circuits *(U108)* and Applications of Matrix Methods: Analysis of Linear Circuits *(U112)*

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FOOD SERVICE MANAGEMENT (U105)

AND

APPLICATIONS OF MATRIX METHODS: FOOD SERVICE AND DIETARY REQUIREMENTS (U109)

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1. FOOD SERVICE MANAGEMENT (U105)

1.1 The Problems of Food Service Management

The food service management for an institution such as a hospital must concern itself with the problems of serving a large number of meals every day. Such a service demands, at the very least: 1) that sufficient quantities of food items be on hand to prepare the menu items, 2) that certain nutrient requirements be met, and 3) that reasonable costs be maintained. There are, of course, other aspects of food service, but for the purposes of this unit we will concentrate on these three.

1.2 Definitions

We define *food items* as purchased raw food, and a *menu item* as a single serving of a dish made from food items. For example, cake is a menu item whose ingredients consist of food items such as eggs, flour, butter and sugar. *Nutrients* are the properties of food such as calories, protein, fat, carbohydrates, calcium, vitamin A.

1.3 Calculating Menu Costs

Let us suppose that we wished to calculate the cost of a menu item, for example, a pound cake. The recipe calls for 1 unit, given as a weight, each of eggs, flour, sugar, and butter. The cost of these units are, respectively, 70, 10, 25, and 50 cents. It is easy to see that the cost of the ingredients in a pound cake is:

$$70(1) + 10(1) + 25(1) + 50(1) = 155 \text{ cents} = \$1.55.$$

Another recipe, one for scrambled eggs, requires $\frac{1}{2}$ unit of eggs and $\frac{1}{4}$ unit of butter. This cost is calculated as:

$$70(\tfrac{1}{2}) + 50(\tfrac{1}{4}) = 47\tfrac{1}{2} \text{ cents.}$$

The cost of each of these menu items is obtained by taking the sum of several products. The products, in each case, are the costs of a unit of food item multiplied by the number of units of that item which is needed in the recipe. This is very simple arithmetic. However, since we are dealing with a large number of recipes with a wide range of ingredients, there is some efficiency in organizing our calculation in a structured way. We will proceed to demonstrate how this may be done.

1.4 A Matrix Representation of Menu Items

The ingredients of each menu item can be arranged as columns of a matrix in the following fashion:

$$\begin{array}{ll|ccccc|}
 & & & & \underline{\text{Menu Item}} & & \\
 & & \text{pound cake} & \text{scrambled eggs} & \text{omelet} & \text{beef strog.}^{*} & \cdots \\
\hline
\text{Food Item} & \text{eggs} & 1 & .5 & 1 & 0 & \cdots \\
 & \text{flour} & 1 & 0 & .25 & 0 & \cdots \\
 & \text{sugar} & 1 & 0 & 0 & 0 & \cdots \\
 & \text{butter} & 1 & .25 & .25 & 0 & \cdots \\
 & \text{beef stroganoff} & 0 & 0 & 0 & 1 & \cdots \\
 & \cdots & . & . & . & . & \cdots \\
 & \cdots & . & . & . & . & \cdots \\
\hline
\end{array}$$

*Beef stroganoff is a convenience food, and is purchased already prepared. For that reason it listed as both a food item and a menu item.

The *columns* of this ingredient matrix contain the quantity of each food item needed for the menu item that is represented by each column. The *rows* of the matrix represent the food items as they appear in various recipes. A zero entry indicates that the food item is not used in a particular recipe. The list of food items and menu items is limited here for simplicity. A practical list would contain hundreds of items.

1.5 The Price Vector

A matrix consisting of a single row or column is called a *vector*. We can arrange the prices of each of the food items in the above matrix as such an ordered row of numbers to define a price vector:

$$\begin{array}{lcccccc} & \text{eggs} & \text{flour} & \text{sugar} & \text{butter} & \text{beef strog.} & \\ \text{Price} = & [70 & 10 & 25 & 50 & 100 & \ldots] \end{array}$$

1.6 Calculating the Cost Vector

If we multiply the price vector by the ingredient matrix, the result is a vector whose entries are the costs for each recipe. That is, for the example values given:

$$[70 \quad 10 \quad 25 \quad 50 \quad 100] \begin{bmatrix} 1 & .5 & 1 & 0 \\ 1 & 0 & .25 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & .25 & .25 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [155 \quad 47.5 \quad 85 \quad 100].$$

Notice the order in which we write the left side of this equation: row vector first, ingredient matrix second.

Compare the computation made previously for the first two of the menu items in this matrix with the computation which is made in matrix multiplication. We can see that the computations are the same, and that the resulting vector does indeed contain, in order, the cost of each menu item.

Some of the advantages of structuring the problem this way should be evident. For one, matrix multiplication accomplished the needed computation. In this form, the data can be easily entered and the operations performed on a computer. Second, a simple change in price can be entered once and will always be applied to all recipes. Finally, new recipes and ingredients can be added to an expanded matrix with little trouble.

Note: Sometimes it is convenient to use matrix multiplication just to sum a series of columns (or rows) of a matrix. For example, suppose we wished to add each column of this matrix:

$$\begin{bmatrix} 2.5 & 10.0 & 9.4 & 7.4 \\ 7.3 & 7.6 & 3.6 & 2.1 \\ 7.0 & 3.2 & 1.9 & 8.3 \end{bmatrix}$$

Verify that the addition could be accomplished by premultiplying the matrix by a row vector, [1 1 1]. This is a convenient way to find the column sums when we are using a computer to perform the calculations.

Question: If we wished to find the sum of the rows of a matrix by means of matrix multiplications, how could we do it?

1.7 Exercises

1. Suppose the number of servings of each recipe to be prepared on a given day for pound cake, scrambled eggs, omelet, and beef stroganoff is, respectively, 5, 10, 2, and 50. Using matrix multiplication, calculate the amount of each food needed on that day. The resulting vector is called the (input) food package for the day.

2. Find the cost of the food package using the price vector given in Section 1.5.

3. Suppose that number of recipes for several days is as follows:

   Day 1: 5, 10, 2, 50
   Day 2: 2, 3, 3, 6
   Day 3: 1, 1, 10, 0.

   These recipes are the same as those in Problem 1. Formulate these needs as a matrix. Find the total amount of each food needed for each of the three days (or find the input food package for each day).

4. Find the cost of the food package for each of the three days using the price vector given in Section 1.5.

1.8 Sample Menu Items and Costs Per Serving

Sample Menu Items	Cost in Cents to Management per Serving
Dressing and Gravy	
Brown Gravy	0.66
Onion Gravy	0.93
Spanish Sauce	3.75
Tartar Sauce	1.98
Bar-B-Q Sauce	1.25
Chicken Gravy	0.53
Cole Slaw Vinegar Dressing	0.36
Pimento Cheese	4.38
Russian Dressing	2.41
French Dressing	1.29
Thousand Island Dressing	1.45
Lemon Sauce	0.76
Whipped Topping	1.08
Pickle Salad Dressing	0.60
Entrees	
Beef Stew with Vegetables	22.95
Chili With Beans	18.46
Meat Loaf and Gravy	22.90
Oven-Fried Steak	28.16
Roast Beef with Gravy	40.93
Country-Style Steak	48.94
Smothered Steak	50.51
Baked Swiss Steak	54.52
Baked Macaroni and Cheese	10.00
Baked Haddock	30.77
Sole Fillet with Tartar Sauce	32.00
Fish Sticks with Tartar Sauce	12.25
Salmon Pattie	52.49
Breaded Pork Chop	56.54
Breaded Pork Cutlet	42.16
Pork Chop with Mexican Sauce	48.30
Deep-Fried Pork Cutlet	42.21
Roast Fresh Ham with Gravy	34.29

Sample Menu Items	Cost in Cents to Management per Serving
Bar-B-Q Chicken	46.21
Fried Chicken	44.50
Roast Turkey with Gravy	38.21
Chicken-Fried Veal Cutlet	44.45
Oven-Fried Veal Cutlet	42.21
Baked Ham Loaf	32.93
Hot Corned Beef	34.50
Smothered Liver with Onions	34.36
Polish Sausage	30.12
Sauteed Chicken Livers	16.80
Vegetables	
Buttered Whole Kernel Corn	8.31
Seasoned Hominy	4.18
Baked Beans	7.90
Seasoned Blackeyed Peas	7.85
Buttered Green Lima Beans	5.34
Buttered Egg Noodles	2.23
Potatoes Au Gratin	4.97
Baked Potato	8.26
Buttered Diced Potatoes	4.16
Creamed Diced Potatoes	2.47
Hash Browned Potatoes	8.49
French Fried Potatoes	16.12
Whipped Potatoes	2.51
Oven-Browned Potato	8.68
Paprika Diced Potatoes	2.84
Buttered Steamed Rice	2.41
Rice Pilaf	3.06
Glazed Sweet Potatoes	8.67
Buttered Steamed Cabbage	5.16
Harvard Beets	2.61
Cauliflower Au Gratin	6.94
Buttered Broccoli	7.55
Buttered Brussels Sprouts	14.29
Buttered Cauliflower	10.30

Sample Menu Items	Cost in Cents to Management per Serving
Buttered Diced Carrots	4.58
Parsley-Buttered Carrots	4.48
Buttered Spinach	6.30
Buttered Chopped Turnip Greens	4.75
Buttered French-Cut Green Beans	8.68
Buttered Canned Green Beans	12.14
Buttered Mixed Vegetables	8.50
Buttered Green Peas	10.26
Buttered Wax Beans	10.28
Stewed Tomatoes	6.47
Buttered Onions	6.82
Seasoned Yellow Squash	12.86
Salads	
Pineapple Waldorf Salad	8.86
Strawberry Jello with Bananas	4.43
Cabbage Slaw/Green Peppers	4.90
Mardi Gras Cole Slaw	5.34
Carrot-Celery Sticks	3.24
Relishes (Crts, Dll Pkls, Rp Olives)	7.52
Stuffed Celery	4.70
Deviled Egg Salad	7.87
Lettuce Wedge/Russian Dressing	6.99
Tossed Salad/French Dressing	6.85
Lettuce Wedge/Salad Dressing	10.93
Tossed Salad/1000 Isle Dressing	5.95
Marinated Vegetable Salad	8.51
Ambrosia Salad	16.29
Jellied Grapefruit Salad	6.98
Peach-Cottage Cheese Salad	10.88
Jellied Pear Salad	8.22
Pineapple-Cheddar Cheese Salad	10.37
Sliced Tomato Salad	10.95
Perfection Salad	4.06

Sample Menu Items	Cost in Cents to Management per Serving
Desserts	
Apple Betty	10.83
Canned Apricots	6.22
Banana Layer Cake	16.75
Gingerbread with Lemon Sauce	10.67
White Layer Cake with Icing	10.57
White Bread	2.36
Milk	8.00
Cherry Pie	13.03
Angel Cake/Whipped Choc Topping	10.50
Gelatin Cubes	2.30
Lemon Sponge Custard	4.12
Canned Fruit Cocktail	9.52
Peach Pie	12.96
Canned Pears	6.72
Vanilla Ice Cream	8.07
Rice Custard	4.22
Pumpkin Pie	15.00

1.9 Model Exam

1. Use Computer Program 1 in Appendix A to calculate AB for

$$A=\begin{bmatrix} 13 & 10.4 & 7.7 \\ 19 & 16.4 & 13.7 \\ 25.4 & 22.4 & 20.2 \\ 32 & 29.4 & 26.7 \\ 37.6 & 35 & 32.4 \\ 42.4 & 40 & 37.2 \\ 46.6 & 44 & 41.3 \end{bmatrix} \quad B=\begin{bmatrix} 34.6 & 5.1 & 11.1 & 17.5 & 24.1 & 29.8 & 39 \\ 32 & 2.5 & 8.5 & 15 & 21.5 & 27 & 36 \\ 0 & 6 & 12 & 18 & 24 & 30 & 36 \end{bmatrix}$$

2. Use the following format

$$\begin{array}{l} \\ \text{Menu 1} \\ \text{Menu 2} \\ \text{Menu 3} \\ \text{Menu 4} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Meat} & \text{Potato} & \text{Vegetable} & \text{Salad} & \text{Dessert} \end{array} \\ \left[\begin{array}{ccccc} \quad & \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad & \quad \end{array}\right] \end{array}$$

to write a matrix of costs, A, for the items shown in the following four menus. Costs per serving may be taken from Section 1.8.

Menu 1

Baked Swiss Steak
Hash Brown Potatoes
Buttered Spinach
Tossed Salad/French Dressing
Canned Fruit Cocktail

Menu 2

Baked Ham Loaf
Glazed Sweet Potatoes
Stewed Tomatoes
Pineapple Waldorf Salad
Gelatin Cubes

Menu 3

Bar-B-Q Chicken
Whipped Potatoes
Harvard Beets
Lettuce Wedge/Russian Dressing
Lemon Sponge Custard

Menu 4

Hot Corned Beef
Oven-Browned Potato
Buttered Steamed Cabbage
Relishes
Cherry Pie

3. Solve this problem using the 4 menus shown above: A cafeteria serves 50 Menu 1, 75 Menu 2, 37 Menu 3, and 46 Menu 4 orders. Use matrix multiplication to find the total cost for each type of menu.

2. APPLICATIONS OF MATRIX METHODS: FOOD SERVICE AND DIETARY REQUIREMENTS (U109)

2.1 Challenge Problem

The manager of food services for a large hospital has the use of a computer to assist him in the management of his service. He understands that a computer is useful in financial accounting, but he is hopeful that it can be used to help him plan ahead and, perhaps, improve his services. One of his problems is that he must keep the cost of food served at a reasonable price and, at the same time, meet the nutritional requirements for balanced or therapeutic diets. Dietitians, of course, plan meals, but the coordination of this activity with the purchasing and preparation of food is needed also. He feels that some of this work could be reduced by using a computer to give information which would help in making decisions.

The computation of the nutrient content of a menu item is simple arithmetic once the data has been assembled. The sum of the products of the number of units of the food items needed for the recipe of the menu item and the nutrients, will, of course, give the total for each nutrient in the menu item. But, there are hundreds of menu items and food items; moreover, new recipes are constantly being added and old ones discarded while the price of food items change. There must be some orderly way of managing all this.

It has been suggested to the manager that he should set up files in matrix format to handle all these data. He is confused about this concept, and he calls you in as an assistant and mathematician to the project. Your task is to help him formulate his problem mathematically, and to set up the files as matrices.

Question: What is meant by setting up files in matrix format?

Answer: Data in the files are arranged in rows and columns so that matrix operations may be applied to them.

2.2 A Recipe Matrix

A recipe matrix is one of the first files constructed. In order to make this clear to the manager, you construct a small matrix of menu items showing their ingredients. With the help of someone such as the food preparation supervisor, you create the sample *recipe matrix*, as you did in Section 1.4 of Unit 105.

$$\begin{array}{cc} & \text{Menu Items} \\ \text{Food Ingredients} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \end{array}$$

2.3 Assignments (Optional)

1. Make a limited recipe matrix. Consult a competent resource person for actual recipes, if needed.

Questions: What does each column of the matrix represent? What does each row of the matrix represent? What does each element of the matrix represent?

Since you will be using this matrix in arithmetic operations, it is necessary that the numbers be in standard units. In quantity food preparation, food is measured by weight rather than spoonfuls, cups, etc. The number of units in the recipe for each food should be sufficient to produce one serving for a normal diet.

2. Create a *nutrient matrix* for each food in your recipe file. Suggestion: Let the entries in column i represent the nutrients in food ingredient i. (See Suggested Support Materials, inside front cover.) Many books on nutrition contain extended tables of nutrient values, for example, *Nutrition*, by Chaney and Ross, Houghton Mifflin, 1971,

contains such a table. The standard nutrients to use are:

1. Food energy (calories)
2. Protein (grams)
3. Fat (grams)
4. Carbohydrates (grams)
5. Ash (grams)
6. Calcium (milligrams)
7. Phosphorous (milligrams)
8. Iron (milligrams)
9. Sodium (milligrams)
10. Potassium (milligrams)
11. Vitamin A (international units)
12. Thiamin (milligrams)
13. Riboflavin (milligrams)
14. Niacin (milligrams)
15. Ascorbic acid (milligrams)

Questions: What do the columns of the nutrient matrix represent?
What do the rows of the nutrient matrix represent?
What does each element of the nutrient matrix represent?

3. Using the recipe matrix and the nutrient matrix, find the nutrient content of each recipe.

Note that if, in constructing the nutrient matrix, you failed to make it conformable for matrix multiplication, either pre- or post-, with the recipe matrix, an adjustment will have to be made.

4. Prepare a price vector (a matrix consisting of one row or one column) for the foods in your recipe matrix. Make your prices conform to "reality" as much as possible, keeping in mind that the hospital may have a reduced price for some items which are purchased in quantity.

Find the cost of each recipe using the recipe matrix and the price vector. Which recipes are the most expensive?

Referring to Assignment 3, compare the nutrient content of the most expensive recipes with that of the least expensive.

5. Prepare a serving matrix, that is, a matrix consisting of the number of servings of each menu item per day for some period of time. One week would be a satisfactory period Keep in mind that not every recipe need be prepared every day. Using this information, and the previous matrices which you constructed, how would you find a matrix showing the amount of each food needed per day? How would you find the cost per day of the food required? Perform these calculations.

2.4 Diets Meeting Certain Requirements

At this point, we turn our attention to a different type of problem. Suppose that a doctor has ordered that a patient's diet meet a minimum daily requirement of 1 unit of thiamin, 2 units of niacin, and 3 units of iron. If we select three menu items (food items could also be used) which contain these vitamins in the following quantities,

$$\begin{array}{l|ccc|} & M_1 & M_2 & M_3 \\ \text{Thiamin} & 1 & 0 & 1 \\ \text{Niacin} & 0 & 2 & 3 \\ \text{Iron} & 4 & 0 & 1 \end{array}$$

what portion of each recipe (or food) should be served to the patient to meet the minimum requirements?

Solution:

Let x_1 = the portion of the first item needed,
x_2 = the portion of the second item needed,
x_3 = the portion of the third item needed.

Then for thiamin $x_1 + x_3 \geq 1$,
for niacin $2x_2 + 3x_3 \geq 2$,
and iron $4x_1 + x_3 \geq 3$.

These conditions in themselves are not enough to guarantee that a triple of values (x_1, x_2, x_3) will give the proportions that the patient needs. This is because the system of inequalities may have solutions in which one or more of the variables takes on a negative value. For example, the triple of values $x_1 = 2$, $x_2 = 3$, $x_3 = -1$ satisfies each of the inequalities. However, we can eliminate the possibility of such solutions by requiring the variables to satisfy the additional three inequalities $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. These inequalities merely formulate the obvious statement that menu portions are never negative.

Now, we can certainly satisfy all six of our inequalities with huge values of x_1, x_2, and x_3. But to meet the patient's dietary requirements more economically, we look for non-negative solutions of the equalities

This suggests the following system of equations:

$$\begin{aligned} x_1 + x_3 &= 1 \\ 2x_2 + 3x_3 &= 2 \\ 4x_1 + x_3 &= 3. \end{aligned}$$

The matrix representation of this system is

$$\begin{array}{ccc} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} & = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ A & X & = B \end{array}$$

Verify that this is true by multiplying A and X, and showing that the product equals B. The results should be the three original equations.

To complete the solution, let A^{-1} be the inverse* of matrix A (if there is an inverse for A).

---

*It is to be remembered that not all matrices have inverses. The inverse exists if and only if $|A|$ (determinant of A) has non zero value. Also a matrix B is called an inverse of A if $BA = I = AB$. Indeed, if the inverse exists then it is unique. For more details see any standard book on linear algebra.

Then $$A^{-1}AX = A^{-1}B$$

and $$IX = A^{-1}B$$
$$X = A^{-1}B.$$

If

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ -2 & \frac{1}{2} & \frac{1}{2} \\ \frac{4}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

then

$$X = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}.$$

2.5 Research Project

Find the recommended daily nutirtional requirements for a normal person. You can use yourself as the person in question in order to specify characteristics such as sex and weight. Construct a recipe matrix of the menu items that you eat for a given week. Does your diet meet the recommended daily nutritional requirements most days? Does the average of the daily intake for a week meet these requirements?

2.6 Model Exam

General Directions:

In this test, do not perform any computations or assign numerical values to the elements of the matrices you construct. To indicate the structure of a matrix, simply label the contents of the columns and rows as shown below.

$$P = \begin{array}{cc} \text{Name of Items Represented by the Columns} & \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} & \text{Name of Items Represented by the Rows} \end{array}$$

1. In food preparation, a *subassembly* is a part of a menu item which may be prepared separately, for example, salad dressing, frosting for a cake, or pie crust dough. Set up a matrix to show the food content of a set of subassemblies.

2. Suppose you wished to compute the cost of each subassembly. What sort of matrix would you need, and what kind of information should it contain? Show the matrix operation which would compute this cost. Represent the matrices as indicated in the general directions.

3. Show how you would compute the nutrient content of the subassemblies.

4. Below are two nonsense matrices called A and B.

$$A = \begin{array}{cc} \text{Gooies} & \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} & \text{Fudge Items} \end{array}$$

Unit cost of fudge items:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix}$$

Interpret the product BA.

5. In this section we have been concerned with the following activities:
   a. Deciding on the kinds of information needed to answer certain questions
   b. Collecting data
   c. Displaying data in a matrix
   d. Deciding on the matrix operations to be applied to obtain information from data
   e. Making computations
   f. Interpreting results

   Which of these activities do you consider the most difficult and which the easiest? Give reasons for your answers. (This question is evaluated on the reasons you give for your answer.)

The Project would like to thank Kenneth R. Rebman of California State University, Hayward, and George Springer of Indiana University, Bloomington, for their reviews, and all others who assisted in the production of this unit.

This unit was field-tested and/or student reviewed by Ellen Cunninghan, SP, St. Mary-of-the-Woods College, St. Mary-of the-Woods, Indiana; Robert M. Thrall, Rice University, Houston, Texas; Philip D. Straffin, Jr., Beloit College, Beloit, Wisconsin; Michael A. Grajek, Hiram College, Hiram, Ohio; Donald G. Beane, The College of Wooster, Wooster, Ohio; Dina Ng, California Polytech State University, San Luis Obispo, California, and; Henry J. Osner, Modesto Junior College, Modesto, California, and has been revised on the basis of data received from these sites.

3. . ANSWERS TO EXERCISES (U105)

Answer to question on page 4.

Postmultiply the matrix by a column vector consisting of 1's.

In the above case, use: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

1. $$\begin{bmatrix} 1 & .5 & 1 & 0 \\ 1 & 0 & .25 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & .25 & .25 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 10 \\ 2 \\ 50 \end{bmatrix} = \begin{bmatrix} 12 \\ 5.5 \\ 5 \\ 8 \\ 50 \end{bmatrix} \begin{matrix} \text{eggs} \\ \text{flour} \\ \text{sugar} \\ \text{butter} \\ \text{beef stroganoff} \end{matrix}$$

Note that it was necessary to post-multiply the ingredient matrix by the number of recipes vector. This vector was formulated as a column vector. Verify that the proper sums of products were formed in order to give the total quantities of eggs, flour, etc.

2. $$[70 \quad 10 \quad 25 \quad 50 \quad 100] \begin{bmatrix} 12 \\ 5.5 \\ 5 \\ 8 \\ 50 \end{bmatrix} = 6420.$$

3. $$\begin{bmatrix} 1 & .5 & 1 & 0 \\ 1 & 0 & .25 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & .25 & .25 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & 1 \\ 10 & 3 & 1 \\ 2 & 3 & 10 \\ 50 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 6.5 & 11.5 \\ 5.5 & 2.75 & 3.5 \\ 5 & 2 & 1 \\ 8 & 3.5 & 3.75 \\ 50 & 6 & 0 \end{bmatrix}$$

4. $$[70 \quad 10 \quad 25 \quad 50 \quad 100] \times \begin{bmatrix} 12 & 6.5 & 11.5 \\ 5.5 & 2.75 & 3.5 \\ 5 & 2 & 1 \\ 8 & 3.5 & 3.75 \\ 50 & 6 & 0 \end{bmatrix} = [6420 \quad 1307.5 \quad 1052.5]$$

4. ANSWERS TO MODEL EXAM (U105)

1.

```
Input Matrices
13.000   10.400    7.700
19.000   16.400   13.700
25.400   22.400   20.200
32.000   29.400   26.700
37.600   35.000   32.400
42.400   40.000   37.200
46.600   44.000   41.300

34.600    5.100   11.100   17.500   24.100   29.800   39.000
32.000    2.500    8.500   15.000   21.500   27.000   36.000
 0.000    6.000   12.000   18.000   24.000   30.000   36.000
Matrix Product
 782.599 138.500  325.099  522.099   721.699  899.199  1158.600
1182.199 220.099  514.699  825.099  1139.300 1420.000  1824.600
1595.639 306.739  714.739 1144.099  1578.539 1967.719  2524.199
2048.000 396.899  925.499 1481.599  2044.099 2548.399  3267.600
2420.959 473.659 1103.659 1766.199  2436.259 3037.479  3892.799
2747.040 539.439 1257.039 2011.599  2774.639 3459.519  4432.799
3020.360 595.459 1386.859 2218.899  3060.259 3816.679  4888.200
```

2.

$$\begin{array}{c|ccccc|} & \text{Meat} & \text{Potatoes} & \text{Vegetables} & \text{Salad} & \text{Dessert} \\ \text{M1} & 54.52 & 8.49 & 6.30 & 6.85 & 9.52 \\ \text{M2} & 32.93 & 8.67 & 6.47 & 8.86 & 2.30 \\ \text{M3} & 46.21 & 2.51 & 2.61 & 6.99 & 4.12 \\ \text{M4} & 34.50 & 8.68 & 5.16 & 7.52 & 13.03 \end{array}$$

3. To find the cost of one serving of an individual menu, we need to sum the costs of the component menu items: i.e., we need to find the sum of a row of the matrix of costs, A. This can be done by post-multiplying A by the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The result will be a vector whose components are the costs of the menus:

[cost (menu 1) cost (menu 2) cost (menu 3) cost (menu 4)]

To find the total cost to the cafeteria, multiply this vector by the number-of-servings vector

$$C = \begin{bmatrix} 50 \\ 75 \\ 37 \\ 46 \end{bmatrix}$$

Notice pre-multiplying matrix A by C gives a cost vector whose elements would represent the total cost for all the menus of:

[meat potatoes vegetables salad dessert]

but not the cost of the individual menus. We see that it is important to consider carefully what the entries in a product matrix represent and the order of multiplication.

5. ANSWERS TO MODEL EXAM (U109)

1-2. Let f_1, f_2, f_3, . . . represent quantities of food items, and S_1, S_2, S_3, . . . be sub-assemblies containing these items. Let c_1, c_2, c_3, . . . be unit costs of these food items. Then

$$\underset{\text{(Cost vector)}}{[c_1 \quad c_2 \quad c_3 \ldots]} \times \underset{\text{(Food item matrix)}}{\begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & \ldots\ldots \end{array} \\ \left[\begin{array}{cccc} f_{11} & f_{12} & f_{13} & \ldots\ldots \\ f_{21} & f_{22} & f_{23} & \ldots\ldots \\ f_{31} & f_{32} & f_{33} & \ldots\ldots \\ . & . & . & \ldots\ldots \end{array}\right] \end{array}} = \text{Cost of each sub-assembly}$$

Some f_{ij} may be zero.

3. Be careful of this problem. It is deliberately vague. One interpretation would be to compute the total nutrient content of several sub-assemblies. Then the problem may be set up as follows:

$$\underset{\substack{\text{(quantity of each}\\ \text{sub-assembly}\\ \text{prepared)}}}{[a_1 \quad a_2 \quad a_3 \ldots]} \quad \underset{\substack{\text{(Nutrient vectors showing}\\ \text{unit quantities of each}\\ \text{for food items.)}}}{\begin{array}{c|cccc} & n_1 & n_2 & n_3 & \ldots\ldots \\ \hline f_1 & x_{11} & x_{12} & x_{13} & \ldots\ldots \\ f_2 & x_{21} & x_{22} & x_{23} & \ldots\ldots \\ f_3 & x_{31} & x_{32} & x_{33} & \ldots\ldots \\ . & . & . & . & \ldots\ldots \end{array}} = \substack{\text{Total of each}\\ \text{nutrient in}\\ \text{set of sub-}\\ \text{assemblies}}$$

4. The product B × A = a cost vector for Gooies.

5. Individual answers are acceptable. However, for the type of problem we are solving here activity a and activity c are probably the critical ones. Activity f is important, but should not be difficult if the others are performed properly. Computations should not be considered difficult if a computer is used. Deciding on the matrix operations to be applied should follow from the matrix structure.

APPENDIX A

```
C*****APPLICATIONS OF MATRIX METHODS - PROGRAM 1
C*****THIS PROGRAM PERFORMS MATRIX MULTIPLICATION ON ANY NUMBER OF PAIRS
      OF MATRICES
      DIMENSION A(20,20),B(20,20), C(20,20),IHEAD(40)
C*****NR = LOGICAL NUMBER FOR THE CARD READER.
      NR = 2
C*****NP = THE LOGICAL NUMBER FOR THE PRINTER.
      NP = 5
C*****READ IDENTIFICATION CARD.  LAST CARD SHOULD CONTAIN /* IN COL. 1-2.
100   READ(NR,1,END=70) IHEAD
1     FORMAT(40A2)
      WRITE(NP,3) IHEAD
3     FORMAT(1H1,40A2//)
C     READ THE NUMBER OF ROWS AND COLUMNS IN THE FIRST MATRIX.
      READ(NR,5) NROW, NCOL
5     FORMAT(215)
C*****READ THE FIRST MATRIX.
      DO 10 I = 1,NROW
      READ(NR,7) (A(I,J),J=1,NCOL)
7     FORMAT(10F5.0)
10    CONTINUE
C*****READ THE NUMBER OF ROWS AND COLUMNS IN THE SECOND MATRIX.
      READ(NR,5) MROW,MCOL
      DO 20 I = 1,MROW
C*****READ THE SECOND MATRIX.
20    READ(NR,7) (B(I,J),J = 1,MCOL)
C*****CALL THE SUBROUTINE TO MULTIPLY THE MATRICES.
      CALL MATMY(A,NROW,NCOL,B,MROW,MCOL,C)
C*****WRITE THE INPUT MATRICES.
      WRITE(NP,25)
25    FORMAT(5X,'INPUT MATRICES'//)
      DO 35 I = 1,NROW
      WRITE(NP,30) (A(I,J),J = 1,NCOL)
30    FORMAT(5X,10F10.3)
35    CONTINUE
      WRITE(NP,40)
40    FORMAT(///)
      DO 50 I = 1,MROW
50    WRITE(NP,30) (B(I,J),J = 1,MCOL)
C*****WRITE THE MATRIX PRODUCT.
      WRITE(NP,55)
55    FORMAT(///5X,'MATRIX PRODUCT'//)
      DO 60 I = 1,NROW
60    WRITE(NP,30) (C(I,J),J=1,MCOL)
C*****RETURN FOR A NEW PROBLEM.
      GO TO 100
70    CALL EXIT
```

```
      SUBROUTINE MATMY(A,NROW,NCOL,B,MROW,MCOL,C)
      DIMENSION A(20,20),B(20,20),C(20,20)
      NP = 5
      IF(NCOL-MROW)20,10,20
20    WRITE(NP,15)
15    FORMAT(5X,'NUMBER OF COLUMNS IN FIRST MATRIX MUST EQUAL NUMBER OF
     1ROWS IN SECOND MATRIX')
      CALL EXIT
10    DO 40 I = 1,NROW
      DO 40 J = 1,MCOL
      C(I,J) = 0.0
      DO 40 K = 1,NCOL
40    C(I,J) = C(I,J) + A(I,K)*B(K,J)
      RETURN
      END
```

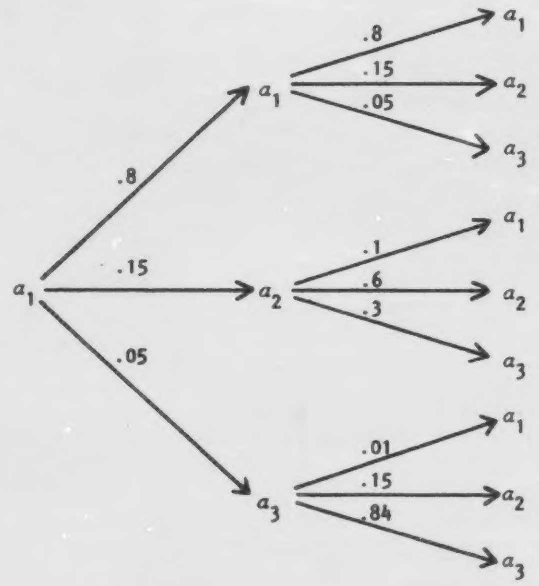
umap

UNITS 107 & 111

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

MARKOV CHAINS (U107) and APPLICATIONS OF MATRIX METHODS: FIXED POINT AND ABSORBING MARKOV CHAINS (U 111)

by Sister Mary K. Keller



APPLICATIONS OF MATRIX METHODS

Units 105 - 112

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MARKOV CHAINS (UNIT 107)
AND
APPLICATIONS OF MATRIX METHODS:
FIXED-POINT AND ABSORBING MARKOV CHAINS (UNIT 111)

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APPENDIX A 30

Intermodular Description Sheet: UMAP Units 107 and 111

Title: MARKOV CHAINS (U107) and APPLICATIONS OF MATRIX METHODS: FIXED POINT AND ABSORBING MARKOV CHAINS (U111)

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Review Stage/Date: III 6/1/78

Classification: APPL MATRIX/MARKOV CHAINS

Suggested Support Material:

References:
Kemeny et al., Finite Mathematics, 2nd Edition, Prentice Hall.

Prerequisite Skills:
1. Elementary notion of probability, probability of one event following another, mutually exclusive events.
2. General concept of a matrix and a probability vector.
3. Be familiar with matrix multiplication, solving a system of linear equations, and raising a matrix to a power.

Output Skills:
1. Be able to define a Markov chain.
2. Be able to interpret powers of matrices representing Markov chains.
3. Be able to recognize certain processes as Markov chains.
4. Be able to draw a tree diagram for a given Markov chain.
5. Be able to formulate a matrix of transition probabilities from a tree diagram of a Markov chain.
6. Be able to make long term predictions using fixed-probability vectors for regular Markov chains.
7. Be able to calculate a fixed-probability vector for regular Markov chains.
8. Be able to recognize an absorbing Markov chain.
9. Be able to structure an absorbing Markov chain into a standard form.
10. Be able to calculate the average time a process will be in each nonabsorbing state.
11. Be able to calculate the probability that a process will end up in a given absorbing state.
12. Be able to calculate how long, on the average, it will take for a process to be absorbed.

Other Related Units:

Food Service Management *(Unit 105)* and Applications of Matrix Methods: Food Service and Dietary Requirements *(Unit 109)*
Computer Graphics *(Unit 106)* and Applications of Matrix Methods: Three Dimensional Computer Graphics and Projections *(Unit 110)*
Electrical Circuits *(Unit 108)* and Applications of Matrix Methods: Analysis of Linear Circuits *(Unit 112)*

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MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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1. MARKOV CHAINS (UNIT 107)

1.1 Introduction

In order to understand what is meant by a Markov chain, consider the following situation.

In a certain class, a teacher has observed that students' performance on tests is affected by how well or poorly they have done on the last test taken. In particular, 80% of the students who did well on the last test will rate well on the next one, 15% will be average, and only 5% will be poor. For those who were rated as average on a test, 60% will continue to be average on the next test, while 10% will do well and 30% poorly. For those students who were rated as poor, only 1% will do well, 15% average, and the remaining 84% will continue to rate low in the next test. We can think of this as a process which will continue through several tests. For the sake of discussion, we will ignore any factor which might upset these predictions.

1.2 Tree Diagrams

These probabilities can be represented by a *tree diagram*. Let the ratings be labeled as:

a_1 = good

a_2 = average

a_3 = poor

where a_1, a_2, a_3 represent the current test score for any student. If we start with a student who has received a good grade, we can show the possibilities for the next test with a diagram like the one below.

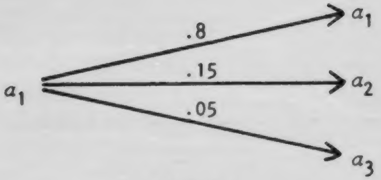


Figure 1.

In Figure 1, the lines drawn from a_1, or branches from a_1, are labeled with the probabilities that any one of them will lead to the next event; that is, getting a grade of good, average, or poor. Since these are the only possibilities, one of them must happen if the student takes another test. For this reason, the sum of the probabilities stemming from any one point must equal 1. Otherwise, some event could happen which is not accounted for.

Since we know the probabilities for the students who receive average or poor grades, we can extend the tree in Figure 1 to show this information.

Figure 2 shows the probabilities through a series of tests. The branches stemming from the left-most a_1 point to the three outcome points for test 2. The branches from each of these three points indicate the probabilities for test 3.

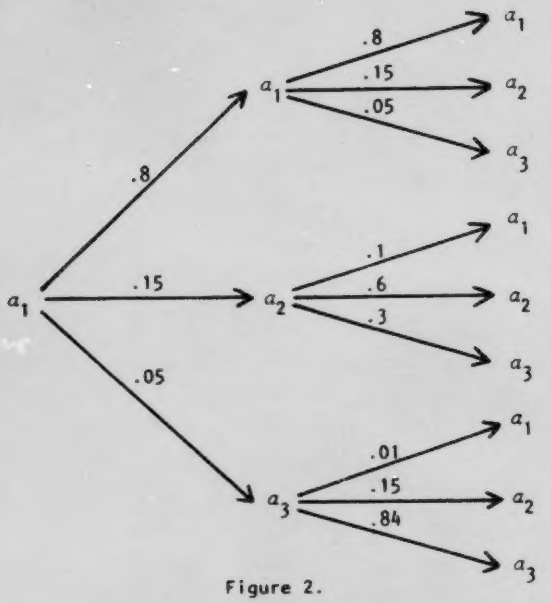


Figure 2.

Representation of probabilities in the form above is called a *tree diagram.*

1.3 Calculating Probabilities From a Tree Diagram

Suppose we wished to know the probability of getting a good rating on the third test for a student who had received a good rating on the first test. If we examine Figure 2, we see that a_1 appears three times in the right-most column, which indicates the outcomes for the third test. These three paths are along the branches:

$$a_1 \rightarrow a_1 \rightarrow a_1$$
$$a_1 \rightarrow a_2 \rightarrow a_1$$
$$a_1 \rightarrow a_3 \rightarrow a_1$$

According to the rules of compound probability, the probability of one event following another is the product of their probabilities.* Therefore, the probability for each of these series of events is

$$a_1 \rightarrow a_1 \rightarrow a_1 = .8(.8) = .64$$
$$a_1 \rightarrow a_2 \rightarrow a_1 = .15(.1) = .015$$
$$a_1 \rightarrow a_3 \rightarrow a_1 = .05(.01) = .0005$$

Since one of these events must occur if the test is taken, and since the events are mutually exclusive (cannot occur together), the probability of receiving a rating of good on the third test is the sum of the probabilities of completing the paths shown above.

Thus, the possibility of reaching a_1 in two moves is $.64 + .015 + .0005 = .6555$. If we were to explore this process beginning at any state, we could compute the probability for any subsequent state in a similar manner.

1.4 The Matrix Representation of a Markov Chain

The format of the computations made in Section 3 suggests that the information in the tree diagram could be structured as a matrix,

$$M = \begin{array}{lccc} & \begin{matrix} a_1 \\ \text{(good)} \end{matrix} & \begin{matrix} a_2 \\ \text{(average)} \end{matrix} & \begin{matrix} a_3 \\ \text{(poor)} \end{matrix} \\ a_1 \text{ (good)} & .8 & .15 & .05 \\ a_2 \text{ (average)} & .1 & .6 & .3 \\ a_3 \text{ (poor)} & .01 & .15 & .84 \end{array}$$

to which we attach the meaning: the probability of going, in one step from

$$a_1 \text{ to } a_1 = p_{11} = .8$$
$$a_1 \text{ to } a_2 = p_{12} = .15$$
$$a_1 \text{ to } a_3 = p_{13} = .05.$$

---

* For the product rule to hold, the events in question must be independent.

Exercise 1:

Complete by giving the meaning of and probability for

_______________ = p_{21} = _______________

_______________ = p_{22} = _______________

_______________ = p_{23} = _______________

_______________ = p_{31} = _______________

_______________ = p_{32} = _______________

_______________ = p_{33} = _______________

The process described in the preceding sections is an example of a *Markov chain*. Such a chain consists of a series of states, and the probabilities of passing to a new one in some defined process. As the example shows, each state is always dependent on the one that precedes it.

1.5 Experiment 1

Use the test example and the computer to compute the probability of being in state a_1 or a_2 or a_3 beginning in any one of the three states after the second test; the third test; the fourth test.

Hint: Do you see that multiplying the matrix M by itself according to the rules of matrix multiplication will give the desired probabilities for the second test?

Interpret $M \cdot M \cdot M = M^3$.

What is the probability that the third test will be rated average if a student was rated as poor in the first test?

Extend the tree diagram in Figure 2 to show the continuation of the process through four steps. Calculate the probabilities of being in state a_1 or a_2 or a_3 after the fourth test, using the tree diagram and the method shown on pages 3 - 4. Compare this with the results obtained by calculating M^4. Does the matrix method produce the same results? Do you see that use of matrices in this problem makes the calculations simpler and more likely to be accurate when using a computer than when working manually from a tree diagram?

1.6 Experiment 2

Assume that women's occupations could be classified as follows:

Housewife, full time = W_1
Housewife, part time work outside = W_2
Full time work outside home = W_3
Full time professional career = W_4

A sample is taken of women who have at least one daughter. The following trends were noted: Of those daughters whose mothers had been full-time housewives, 50% were classified as W_1, 25% as W_2, 20% as W_3, and 5% as W_4. For those whose mothers were housewives and worked part time outside the home, 60% of the daughters also did so but 15% became full-time housewives and 15% worked full time, with 10% having a full professional career. The daughters of the full-time workers were distributed as follows: 20% full-time housewives, 25% part-time workers, 40% full-time workers, and 15% professional women. Finally, the daughters of professional women were distributed in this fashion: 30% housewives, 20% part-time workers, 20% full-time workers, and 30% professional women.

Construct a matrix to represent a Markov chain for these data.

Assuming that this trend continues, find the probabilities that a woman will have the same career as her grandmother.

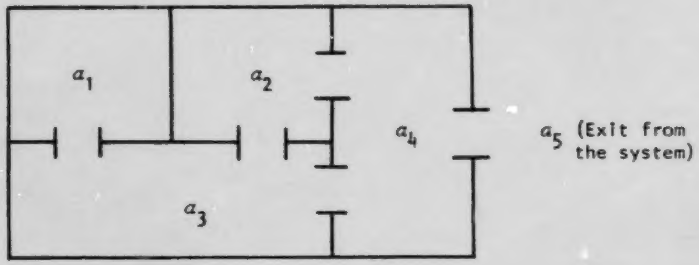
Calculate M^2, M^3, M^4, M^5, M^6 . . . up to any power you wish for this matrix. You are now able to make long term predictions about this process. What seems to be happening?

1.7 Model Exam (Unit 107)

1. In the matrix representation of a Markov chain, what do the elements of the matrix represent?

2. The row sum of any Markov matrix must be 1. Why?

3. The following diagram represents a maze. Each compartment can be considered a "state" of the system. If a rat is placed in compartment a_1, what is the porbability that he will escape from the maze after a given number of trails? (A trial consists of a move from one compartment to another.) Where there is only one way out of a compartment, the probability of choosing that exit is, of course, 1. Clearly, if a compartment cannot be reached directly from another compartment, the probability of passing between these two is zero. Movement when there are multiple exits is considered equally probable.



Rat Maze

Draw a tree diagram to represent this system, and then set up a matrix of possibilities. Compute the probability a rat leaves the maze after three trials.

2. APPLICATIONS OF MATRIX METHODS: FIXED-POINT AND ABSORBING MARKOV CHAINS (UNIT 111)

2.1 Challenge Problem

Competition is a way of life for the producers of many things, from TV shows to detergents. One problem stems from the fickle nature of consumers. They tend to switch from one product to another.

For example, consider three TV networks which are competing for viewers in a given time slot. Three shows, SUNNY DAYS, LOTSA GUNS, and MOON CREATURES are all broadcast on Tuesdays at 7:00 p.m. Surveys taken indicate that for those who watch SUNNY DAYS one week there is a probability that 60% will continue to watch it the next week, while 30% will probably switch to MOON CREATURES, and 10% to LOTSA GUNS. For persons who watch MOON CREATURES there is a 50% chance that they will continue to do so the next week, with 40% changing to LOTSA GUNS, and 10% going to SUNNY DAYS. Finally, those who watch LOTSA GUNS have a probability of 70% of staying with the show the following week, and a 30% probability of switching to MOON CREATURES.

Let us formulate this information as a transition matrix:

$$\begin{array}{c} \\ \text{SD} \\ \text{MC} \\ \text{LG} \end{array} \begin{array}{c} \begin{array}{ccc} \text{SD} & \text{MC} & \text{LG} \end{array} \\ \begin{bmatrix} .6 & .3 & .1 \\ .1 & .5 & .4 \\ .0 & .3 & .7 \end{bmatrix} \end{array}$$

SUNNY DAYS started out with 70% of the audience, MOON CREATURES had 10% and, LOTSA GUNS had 20%. In spite of the good start, the cast of SUNNY DAYS were worrying about their jobs at the end of the fifth week, and were definitely out of a job by the tenth week.

Could this have been predicted? The answer is yes, if it is assumed that the trend shown in the survey continued. In fact, it is possible to predict that eventually SUNNY DAYS will have 9.4% of the viewers, MOON CREATURES will have about 37.5%, and LOTSA GUNS will hold 53.1% of the audience. When that point is reached, there will be no further changes. The *probabilities become fixed.*

In Unit 107 we showed that we could predict the state of a Markov chain as the process went through several stages. We did this my multiplying the matrix by itself, or raising it to a power. We did not explore the possibility of the matrix reaching a *steady state*, that is, that raising the matrix to higher and higher powers no longer changed the probabilities. We consider this situation now.

2.2 Regular Transition Matrices

A transition matrix representing a Markov chain is said to be *regular* if some power of the matrix has only positive components.

The transition matrix from the challenge problem is an example of a regular matrix. Although the original matrix has a zero element, if we take the second power, we find that all of the elements are positive. Verify this by constructing the matrix and using Program 7 in Appendix A to find some power of it.

2.3 Fixed-Probability Vectors

A row vector that consists of non-negative elements whose sum is 1 is called a probability vector. From this definition each single row of a transition matrix is a probability vector. If a transition matrix is regular, then after a number of steps, sometimes a large number, the probability vectors (rows) tend to become the same and remain fixed. To illustrate this, we use the challenge problem. When the transition matrix for TV shows is raised to the 20th power, it becomes

$$\begin{bmatrix} 0.0937 & 0.3749 & 0.5312 \\ 0.0937 & 0.3749 & 0.5312 \\ 0.0937 & 0.3749 & 0.5312 \end{bmatrix}.$$

When this happens the process is in a "steady" state. The probabilities will not change in future steps. The row vector which gives these probabilities,

$$[0.0937 \quad 0.3749 \quad 0.5312]$$

is called the fixed-probability vector.

2.4 Calculating a Fixed-Probability Vector

We can, of course, search for a fixed-probability vector by raising a regular transition matrix to a power, continuing until the fixed state is reached. With a computer this is not particularly difficult, although it may converge slowly and the result be only approximate. There is, however, a direct way of obtaining the fixed-probability vector.

If p is a fixed-probability vector for a matrix A, then it can be shown that $pA = p$. If we use this theorem, we can set up a system of equations which can be solved for the vector p

2.5 Experiment 1

Verify that multiplying the transition matrix for the challenge problem in Section 2.1 by the vector obtained in 2.3 gives the same vector as the product. Thus

$$[0.0937 \quad 0.3749 \quad 0.5312] \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.4 \\ 0.0 & 0.3 & 0.7 \end{bmatrix} = 0.937 \cdot 0.3749 \quad 0.5312.$$

Use Program 1.

2.6 A Fixed Probability Vector from a System of Linear Equations

Since, for the fixed-probability vector p and the regular matrix A, $pA = p$, then, if

$$A = \begin{bmatrix} 0 & 1 \\ .6 & .4 \end{bmatrix}$$

and

$$p = [x_1 \quad x_2] \, ,$$

$$[x_1 \quad x_2] \begin{bmatrix} 0 & 1 \\ .6 & .4 \end{bmatrix} = [x_1 \quad x_2] \, .$$

If we carry out the indicated multiplication, we obtain

$$\begin{bmatrix} .6x_2 & x_1 + .4x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

or

$$.6x_2 = x_1$$

$$x_1 + .4x_2 = x_2$$

and because the sum of any row probability vector must equal 1,

$$x_1 + x_2 = 1.$$

2.7 Experiment 2

Use Program 6 in Appendix A to solve the three equations in two unknowns which were developed in Section 2.6 We restate them as

$$\begin{aligned} -x_1 + .6x_2 &= 0 \\ x_1 - .6x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

Interpret the result.

Show that the same result could be found by raising matrix A to a sufficiently high power. Use Program 7

2.8 Experiment 3

Four companies are competing against each other with products in toothpaste. A survey shows that the shift from one brand to the other can be presented by this transition matrix.

$$\begin{array}{c|cccc} \text{Brand} & A & B & C & D \\ A & .6 & .2 & .1 & .1 \\ B & .1 & .7 & .1 & .1 \\ C & .1 & .1 & .7 & .1 \\ D & .1 & .2 & .1 & .6 \end{array} \, .$$

What is the long term prediction for each company's share of the market?

What change would occur, if any, if company D changed its product and a new survey showed the transition matrix to be:

$$\begin{array}{c|cccc} & A & B & C & D \\ A & .5 & .2 & .1 & .2 \\ B & .1 & .6 & .2 & .1 \\ C & .1 & .1 & .7 & .1 \\ D & .2 & .1 & .1 & .6 \end{array} \, ?$$

2.9 Absorbing Markov Chains

Some Markov chains contain states from which, once entered, departure is no longer possible. This state from which there is no return is called an *absorbing state*. We might have considered the rat maze problem in Unit 107 as having such a state if we assumed that once the rat left the maze it could not go back in.

We can recognize an absorbing state from a transition matrix. Any state, a_i, for which the element a_{ii} is equal to 1 and all other elements of that row are zero is an absorbing state. As an example, recall the rat maze problem on page 7. The transition matrix is

$$\begin{array}{c|ccccc|} & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & 0 & 0 & 1 & 0 & 0 \\ a_2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ a_3 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ a_4 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ a_5 & 0 & 0 & 0 & 0 & 1 \end{array}.$$

It should be immediately evident that a_5 is an absorbing state. The probability of going from a_5 to a_1, a_2, a_3, or a_4 is zero in each case. We note that a_{55} equals 1 and all the other elements in that row are zero.

For a Markov chain to be an absorbing chain it must be possible to get from any non-absorbing state to an absorbing state. One way to recognize this from the matrix representation of a Markov chain is to examine the columns which contain the 1 for the absorbing states. For each such column the remaining elements must not be all zeros if there is to be a transition to this absorbing state. For example, the following Markov chain contains one absorbing state but is not an absorbing chain.

$$\begin{array}{c|ccccc|} & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & .5 & .4 & 0 & .1 & 0 \\ a_2 & .3 & .2 & 0 & .2 & .3 \\ a_3 & 0 & 0 & 1 & 0 & 0 \\ a_4 & 0 & 0 & 0 & .8 & .2 \\ a_5 & 0 & 1 & 0 & 0 & 0 \end{array}$$

The state a_3 is an absorbing state, but there is no way to reach it from any other state. Since it is the only absorbing state in this chain the chain in not an absorbing chain. We could verify this by drawing the tree diagram for this chain.

2.10 Exercise for Absorbing Markov Chains

State whether the following transition matrices are for absorbing or for nonabsorbing Markov chains. Why?

a. $\begin{bmatrix} 1 & 0 \\ .5 & .5 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ .5 & .5 \end{bmatrix}$

c. $\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$

2.11 A Second Challenge Problem

The Ace Collection Agency decides to add a service for its department store customers, and, perhaps, improve its own business. The president of Ace has observed that some department stores turn over their bad accounts for collection at varying times, while other companies rarely use the agency. The latter companies simply write off unpaid bills after repeated attempts at collecting on their own. The president of Ace proposes that, for a reasonable fee, his agency will analyze the paying habits of customers who have charge accounts with department stores. This analysis will produce, it is claimed, information that will enable a store to decide on a policy for turning over bad accounts to a collection agency. At the same time, the analysis will give the store a way of calculating how long, on the average, it takes for accounts to be either paid up or classified as bad.

The manager of Homer Department Store, after seeing this analysis service advertised, decides to try it, but he insists that the method applied to determining any policies for the store be made clear to him before they are effective. He asks that a representative from Ace give an explanation of how it will be determined that a debt will probably end up as bad, or how long debts are likely to stay in various stages of being overdue.

The Ace representative agrees to give an explanation. He begins with a hypothetical case. Suppose, he says, that after studying your accounts it was found from past history that your customers' paying habits could have probabilities attached to them. These probabilities of changing status from month to month are shown in Table I.

TABLE I

Probabilities of Future Debts of a Typical Customer

		Future Months in Arrears							
		0	1	2	3	4	5	Paid-up	Bad
	0	.60	.15	0.0	0.0	0.0	0.0	.25	0.0
Present Months in Arrears	1	.20	.35	.25	0.0	0.0	0.0	.20	0.0
	2	.10	.20	.10	.27	0.0	0.0	.13	0.0
	3	.05	.10	.20	.18	.37	0.0	.10	0.0
	4	.02	.03	.07	.30	.28	.15	.15	0.0
	5	.01	.04	.04	0.0	.25	.45	.06	.15
	Paid-up	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0
	Bad	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0

In this table, the status of accounts is given in months overdue. If there are only current charges, this appears in the 0 column. The entries in the table are the probabilities of changing status from one month to the next. For example, a customer who is two months in arrears (the row labeled 2) has a probability of .10 of having paid sufficient amounts on his account to be classified as having only current charges next month. The same customer has a probability of .30 of being still 2 months behind in the next month.

Some members of the Homer Department Store had sufficient mathematical training to recognize that this table (Table I) could be considered an absorbing Markov chain. However, they were not advanced enough to know how the questions that were asked by the president could be answered from this information. More explanation was needed.

2.12 Standard Form for an Absorbing Markov Chain

Before answering some of the questions raised, it is necessary to rearrange the absorbing Markov chain to standard form. This requires interchanging some rows and columns of the matrix so that absorbing states are placed first. We illustrate this using a simpler matrix than the one derived from Table I.

Given the absorbing Markov chain in matrix form:

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .8 & 0 & .2 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

we look for the absorbing states. From the discussion in Section 2.9 we should recognize states a_1 and a_4 as absorbing states. To obtain a *standard form* for the matrix, interchange the columns and rows so that there is an identity matrix in the upper left hand corner. As you can see, this can be accomplished by placing the absorbing states first. This does not change any relationship, i.e., the probabilities of going from one state to another are preserved.

$$\begin{array}{c c} & \begin{array}{cccc} a_1 & a_4 & a_2 & a_3 \end{array} \\ \begin{array}{c} a_1 \\ a_4 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .5 & 0 & 0 & .5 \\ 0 & .2 & .8 & 0 \end{bmatrix} \end{array}$$

2.13 Partitioning the Standard Form

Once the matrix is in standard form, we can proceed to *partition* it in such a way that four matrices are formed from the original. Later we will see that we can use these new matrices to help answer our questions about the charge accounts. We partition the matrix in this example so that there is an identity matrix in the upper left hand corner. Thus

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline .5 & 0 & 0 & .5 \\ 0 & .2 & .8 & 0 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} I & O \\ \hline S & T \end{array}\right]$$

and we label each new matrix as follows.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad S = \begin{bmatrix} .5 & 0 \\ 0 & .2 \end{bmatrix} \quad T = \begin{bmatrix} 0 & .5 \\ .8 & 0 \end{bmatrix}$$

If we review the original matrix, we can see that the entries in S are the probabilities of being absorbed, and the entries in T are the probabilities of being in nonabsorbing states. This is true because .5 is the probability of going from a_2 to a_1, and .2 is the probability of going from a_3 to a_4. (a_1 and a_4 are the two absorbing states. Similar statements can be made for the entries in T.)

There is a theorem (which we state but will not prove here) that is useful for our purposes. On the average, the number of times a process will be in each nonabsorbing state can be found by calculating $N = (I - T)^{-1}$ where I is an identity matrix and T is the matrix formed by the partition just made.

Program 8 in Appendix A can be used to calculate $N = (I - T)^{-1}$ for our sample problem. However, since this is a very simple matrix we will do the calculations by hand in order to illustrate the intermediate steps.

$$I - T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & .5 \\ .8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -.5 \\ -.8 & 1 \end{bmatrix}.$$

$$(I - T)^{-1} = \begin{bmatrix} 1 & -.5 \\ -.8 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{10}{6} & \frac{5}{6} \\ \frac{8}{6} & \frac{10}{6} \end{bmatrix}.$$

$$\text{Therefore } N = \begin{array}{c c} & \begin{array}{cc} a_2 & a_3 \end{array} \\ \begin{array}{c} a_2 \\ a_3 \end{array} & \begin{bmatrix} \frac{10}{6} & \frac{5}{6} \\ \frac{8}{6} & \frac{10}{6} \end{bmatrix} \end{array}.$$

2.14 Making Decisions Based on Probability

All the information in a Markov chain consists of probabilities, but in the case of the charge accounts these probabilities were based on the past history of a large number of people's paying habits. They are likely to be fairly predictive of the future. In the absence of any other knowledge, past history forms the best basis for making decisions about future events. We will interpret the entries in matrix N and see how they form a basis for additional information upon which some decisions might be made.

The interpretation of the entries of matrix N is this. Starting in one of the nonabsorbing states, say, a_2, the mean number of times in state a_2 before absorption is $\frac{10}{6}$ and in state a_3 is $\frac{5}{6}$. The total time before absorption for state a_2 is $\frac{10}{6} + \frac{5}{6}$ or $\frac{15}{6}$. A similar interpretation could me made for the other entries.

Question:	If this matrix had come from the charge account problem, how would you interpret the entries?
Answer:	If a customer was in state a_2, we would predict that on the average in 2½ months he would either be paid-up or become a bad account.

The row sums of N give the average time for each state before it is absorbed. If we wish to find the row sums for some larger matrix, using the computer, we can use Program 1 in Appendix A and multiply N by a column matrix consisting of 1's. For example,

$$\begin{bmatrix} \frac{10}{6} & \frac{5}{6} \\ \frac{8}{6} & \frac{10}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{15}{6} \\ \frac{9}{6} \end{bmatrix}.$$

This is a convenient way to calculate row sums for a large matrix. The column matrix should have enough 1's to be conformable for multiplication.

2.15 The Probability of Reaching a Given Absorbing State

We still have the question concerning the probability of a given absorbing state as the final one. If you will accept another theorem, we can answer this question. According to the theorem, the product of N, just computed, and the matrix S from the partition on page 17 is a matrix which gives the probabilities of ending up in a given absorbing state. From our example

$$N = \begin{bmatrix} \frac{10}{6} & \frac{5}{6} \\ \frac{8}{5} & \frac{10}{6} \end{bmatrix} \qquad S = \begin{bmatrix} \frac{5}{10} & 0 \\ 0 & \frac{2}{10} \end{bmatrix}.$$

Then

$$A = NS = \begin{array}{c} \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} a_1 & a_4 \end{array} \\ \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{4}{6} & \frac{2}{6} \end{bmatrix} \end{array}.$$

We interpret the entries in matrix A as follows. Starting in state a_2 there is a probability of $\frac{5}{6}$ of absorption in state a_1, and a probability of $\frac{1}{6}$ of absorption in state a_4. A similar interpretation is made for the other entries.

In the original example of Section 2.12 all of the matrices I, O, S, and T turn out to be square. This will always be true for I and T, but is not generally the case for O and S. Consider the matrix below along with its standard form

$$\begin{array}{c|ccccc} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline a_1 & 1 & 0 & 0 & 0 & 0 \\ a_2 & .5 & 0 & .4 & 0 & .1 \\ a_3 & 0 & .7 & 0 & .2 & .1 \\ a_4 & 0 & .8 & 0 & .2 & 0 \\ a_5 & 0 & 0 & 0 & 0 & 1 \end{array}.$$

The standard form

$$\begin{array}{c|cc|ccc|}
 & a_1 & a_5 & a_2 & a_3 & a_4 \\
\hline
a_1 & 1 & 0 & 0 & 0 & 0 \\
a_5 & 0 & 1 & 0 & 0 & 0 \\
\hline
a_2 & .5 & .1 & 0 & .4 & 0 \\
a_3 & 0 & .1 & .7 & 0 & .2 \\
a_4 & 0 & 0 & .8 & 0 & .2
\end{array}$$

and we label each new matrix as follows:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} .5 & .1 \\ 0 & .1 \\ 0 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & .4 & 0 \\ .7 & 0 & .2 \\ .8 & 0 & .2 \end{bmatrix}.$$

Question: If the matrix A had come from the charge account problem, how would you interpret the entries?

Answer: If a customer was in state a_2, we would predict that there was a probability of 5/6 that he would end up in state a_1 (which might be the state "paid up"), and there was a probability of 1/6 that he would be absorbed in state a_4 ("bad account").

2.16 Experiment 4

Form an absorbing Markov chain from Table 1 on page 15. Put the matrix in standard form and then partition it as shown in Section 2.13. Use Program 8 in Appendix A to calculate the matrix N, and Program 1 to find the row sums of N and the product N × S.

Write a report on the information you can give the Homer Department Store as a result of these computations.

2.17 Model Exam (Unit 111)

1. Show that this matrix is a regular matrix.

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

2. Find a fixed-point for the following matrix.

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

3. If P in question 3 is raised to the 100th power, what is the approximate value of the entry in the first row, first column?

4. Joe, as a student, is not very regular in completing assignments. However, if Joe is late with an assignment on one due date, he is 70% sure to have the next one in on time. If he finishes an assignment on time, there is only a 20% chance that he will finish the next one on time. In the long run, what percent of time does Joe miss due dates for his assignments?

5. Does the following matrix represent an absorbing Markov chain? Give the reason for your answer.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & 0 & .5 \\ 0 & 0 & 1 & 0 \\ 0 & .7 & 0 & .3 \end{bmatrix}$$

6. Put the following absorbing chain in standard form.

$$\begin{array}{c|cccc|}
 & a_1 & a_2 & a_3 & a_4 \\
\hline
a_1 & 1 & 0 & 0 & 0 \\
a_2 & .1 & .2 & .5 & .2 \\
a_3 & .2 & .3 & .3 & .2 \\
a_4 & 0 & 0 & 0 & 1
\end{array}$$

Which are the absorbing states in this chain?

7. If we start in state a_3, how many steps will there be on the average before absorption?

8. What is the probability that if we start in state a_2, absorption will occur in state a_4?

3. ANSWERS TO EXERCISES (UNIT 107)

Exercise 1:

a_2 to a_1 = p_{21} = .1

a_2 to a_2 = p_{22} = .6

a_2 to a_3 = p_{23} = .3

a_3 to a_1 = p_{31} = .01

a_3 to a_2 = p_{32} = .15

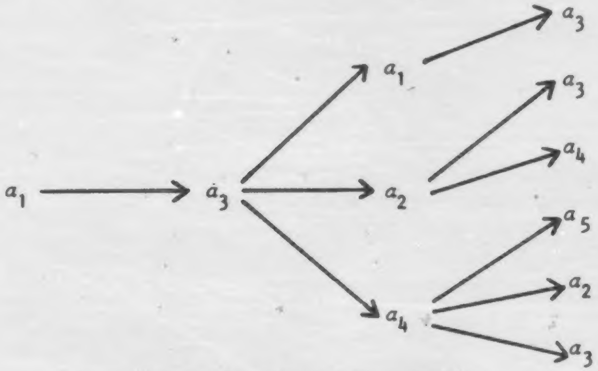
a_3 to a_3 = p_{33} = .84

4. ANSWERS TO MODEL EXAM (UNIT 107)

1. Each element, a_{ij}, represents the probability that a process which starts in the i^{th} state will go to the j^{th} state in one step. j can be equal to i.

2. Each row in a Markov matrix represents the probabilities for all possible next states. The sum of these probabilities must equal 1 to account for all possible states.

3



A tree diagram for the Rat Maze Problem.

The Markov chain is:

$$\begin{array}{c|ccccc|}
 & a_1 & a_2 & a_3 & a_4 & a_5 \\
a_1 & 0 & 0 & 1 & 0 & 0 \\
a_2 & 0 & 0 & 1/2 & 1/2 & 0 \\
a_3 & 1/3 & 1/3 & 0 & 1/3 & 0 \\
a_4 & 0 & 1/3 & 1/3 & 0 & 1/3 \\
a_5 & 0 & 0 & 0 & 0 & 0
\end{array}$$

The probability that the rat leaves the maze in three trials in 1/9. Indeed, from the tree diagram above, the only possibility for the rat to leave the maze is to travel through the branch a_1 --- a_3 --- a_4 --- a_5 . By the compound probability rule, the probability of this event is the product of probabilities from a_1 to a_3, a_3 to a_4, a_4 to a_5. Thus the probability that the rat leaves the maze is $1 \cdot 1/3 \cdot 1/3 = 1/9$.

5. ANSWERS TO EXERCISES (UNIT 111)

		Reason
a.	absorbing	There is an absorbing state, and there is a way to reach it.
b.	nonabsorbing	There is no absorbing state.
c.	absorbing	There is an absorbing state and a way to reach it.
d.	absorbing	There is an absorbing state and a way to reach it.

6. ANSWERS TO MODEL EXAM (UNIT 111)

1.

$$P^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{16} & \frac{9}{16} \end{bmatrix}$$

Since a power of P is positive, the matrix is regular.

2.

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

3. $\frac{2}{3}$

4. Let a_1 represent "assignments on time" probability, and a_2 represent "assignments late" probability. Then

$$\begin{array}{c|cc} & a_1 & a_2 \\ \hline a_1 & .2 & .8 \\ a_2 & .7 & .3 \end{array}$$

is the Markov chain for this problem. Solving for the fixed point for this matrix, we find $a_1 = \frac{7}{15}$, and $a_2 = \frac{8}{15}$. The long run probability that Joe's assignments will be late is 53.333% $(\frac{8}{15})$.

5. The matrix represents an absorbing Markov chain. It has two absorbing states, a_1 and a_3.

6.

$$\begin{array}{c|cccc} & a_1 & a_4 & a_2 & a_3 \\ \hline a_1 & 1 & 0 & 0 & 0 \\ a_4 & 0 & 1 & 0 & 0 \\ a_2 & .1 & .2 & .2 & .5 \\ a_3 & .2 & .2 & .3 & .3 \end{array}$$

States a_1 and a_4 are absorbing.

7.

$$T = \begin{bmatrix} .2 & .5 \\ .3 & .3 \end{bmatrix}$$

$$N = (I - T)^{-1} = \begin{bmatrix} .8 & -.5 \\ -.3 & .7 \end{bmatrix}^{-1} \approx \begin{bmatrix} 1.70 & 1.22 \\ .73 & 1.95 \end{bmatrix}$$

The number of steps before absorption, beginning in a_3 is approximately .73 + 1.95 = 2.68.

8.

$$N \times S \quad \begin{bmatrix} 1.70 & 1.22 \\ .73 & 1.95 \end{bmatrix} \times \begin{bmatrix} .1 & .2 \\ .2 & .2 \end{bmatrix} \approx \begin{bmatrix} .414 & .584 \\ .463 & .586 \end{bmatrix}$$

The probability is 58.4%

Partial Answer for Experiment 4

	Paid-up	Bad	0	1	2	3	4	5
Paid-up	1	0	0	0	0	0	0	0
Bad	0	1	0	0	0	0	0	0
0	.25	0.0	.60	.15	0.0	0.0	0.0	0.0
1	.20	0.0	.20	.35	.25	0.0	0.0	0.0
2	.13	0.0	.10	.20	.30	.27	0.0	0.0
3	.10	0.0	.05	.10	.20	.18	.37	0.0
4	.15	0.0	.02	.03	.07	.30	.28	.15
5	.06	.15	.01	.04	.04	0.0	.25	.45

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} .25 & 0.0 \\ .20 & 0.0 \\ .13 & 0.0 \\ .10 & 0.0 \\ .15 & 0.0 \\ .06 & .15 \end{bmatrix} \qquad T = \begin{bmatrix} .60 & .15 & 0.0 & 0.0 & 0.0 & 0.0 \\ .20 & .35 & .25 & 0.0 & 0.0 & 0.0 \\ .10 & .20 & .30 & .27 & 0.0 & 0.0 \\ .05 & .10 & .20 & .18 & .37 & 0.0 \\ .02 & .03 & .07 & .30 & .28 & .15 \\ .01 & .04 & .04 & 0.0 & .25 & .45 \end{bmatrix}$$

APPENDIX A

PROGRAM 1

```
C*****APPLICATIONS OF MATRIX METHODS - PROGRAM 1
C*****THIS PROGRAM PERFORMS MATRIX MULTIPLICATION ON ANY NUMBER OF PAIRS
C      OF MATRICES
      DIMENSION A(20,20),B(20,20), C(20,20),IHEAD(40)
C*****NR = LOGICAL NUMBER FOR THE CARD READER.
      NR = 2
C*****NP = THE LOGICAL NUMBER FOR THE PRINTER.
      NP = 5
C*****READ IDENTIFICATION CARD.  LAST CARD SHOULD CONTAIN /* IN COL. 1-2
100   READ(NR,1,END=70) IHEAD
1     FORMAT(40A2)
      WRITE(NP,3) IHEAD
3     FORMAT(1H1,40A2//)
C*****READ THE NUMER OF ROWS AND COLUMNS IN THE FIRST MATRIX.
      READ(NR,5) NROW, NCOL
5     FORMAT(2I5)
C*****READ THE FIRST MATRIX.
      DO 10 I = 1,NROW
      READ(NR,7) (A(I,J),J=1,NCOL)
7     FORMAT(10F5.0)
10    CONTINUE
C*****READ THE NUMBER OF ROWS AND COLUMNS IN THE SECOND MATRIX.
      READ(NR,5) MROW,MCOL
      DO 20 I = 1,MROW
C*****READ THE SECOND MATRIX.
20    READ(NR,7) (B(I,J),J = 1,MCOL )
C*****CALL THE SUBROUTINE TO MULTIPLY THE MATRICES.
      CALL MATMY(A,NROW,NCOL,B,MROW,MCOL,C)
C*****WRITE THE INPUT MATRICES.
      WRITE(NP,25)
25    FORMAT(5X,'INPUT MATRICES'//)
      DO 35 I = 1,NROW
      WRITE(NP,30) (A(I,J),J = 1,NCOL)
30    FORMAT(5X,10F10.3)
35    CONTINUE
      WRITE(NP,40)
40    FORMAT(///)
      DO 50 I = 1,MROW
50    WRITE(NP,30) (B(I,J),J = 1,MCOL)
C*****WRITE THE MATRIX PRODUCT.
      WRITE(NP,55)
55    FORMAT(///5X,'MATRIX PRODUCT'//)
      DO 60 I = 1,NROW
60    WRITE(NP,30) (C(I,J),J=1,MCOL)
C*****RETURN FOR A NEW PROBLEM.
      GO TO 100
70    CALL EXIT
```

PROGRAM 6

```
C****APPLICATIONS OF MATRIX METHODS - PROGRAM 6
      DIMENSION A(20,4),B(20,4),C(20,4),IHEAD(40),OBL(20,4)
      NR = 2
      NP = 5
C****READ, PAGE AND WRITE HEADING
      READ(NR,1) IHEAD
1     FORMAT(40A2)
      WRITE(NP,3) IHEAD
3     FORMAT(1H1,40A2//)
C****CREATE OBLIQUE TRANSFORMATION MATRIX FOR PLOTTING.
      DO 4 I = 1,4
      DO 4 J = 1,4
4     OBL(I,J) = 0.0
      OBL(1,1)= 1.0
      OBL(2,2) = 1.0
      OBL(4,4) = 1.0
      OBL(3,1 ) = -0.4333
      OBL(3,2) = -0.2500
C****READ THE NUMBER OF POINTS, MAX = 20.
      READ(NR,5) N
5     FORMAT(I2)
C****READ THE COORDINATES OF THE POINTS ON THE FIGURE.
      DO 10 I = 1,N
      READ(NR,6) (A(I,J),J = 1,4)
6     FORMAT(4F5.0)
10    CONTINUE
C**** THIS TRANSFORMATION IS CARRIED OUT TO GIVE AN OBLIQUE PROJECTION.
C****ALSO, THE Z AXIS IS FORESHORTENED.
      CALL MATMY(A,N,4,OBL,4,4,C)
      WRITE(NP,15)
15    FORMAT(5X,'ORIGINAL FIGURE',//)
      CALL KPLOT(C,N,1.0,0,1)
C*****READ, PAGE AND WRITE HEADING
100   READ(NR,1,END=30) IHEAD
      WRITE(NP,3) IHEAD
C****READ THE TRANSFORMING MATRIX
      DO 20 I = 1,4
      READ(NR,6) (B(I,J),J = 1,4)
20    CONTINUE
      CALL MATMY(A,N,4,B,4,4,C)
      CALL MATMY(C,N,4,OBL,4,4,B)
      CALL KPLOT(B,N,1.,0,1)
      GO TO 100
30    CALL EXIT
```

PROGRAM 7

```
C ***** PROGRAM 7 APPLICATIONS OF MATRIX METHODS
C ***** POWERS OF MATRICES
      DIMENSION A(20,20),B(20,20),C(20,20),IHEAD(40)
      NR = 2
      NP = 5
      READ(NR,5)IHEAD
5     FORMAT(40A2)
      WRITE(NP,6)IHEAD
6     FORMAT(1H1,40A2,///)
C***** READ THE NUMBER OF ROWS AND COLUMNS IN THE MATRIX.
C***** N = THE POWER TO WHICH THE MATRIX IS TO BE RAISED.
      READ(NR,10)NROW,NCOL,N
10    FORMAT(3I5)
      DO 20 I = 1,NROW
      READ(NR,15) (A(I,J),J=1,NCOL)
15    FORMAT(16F5.0)
20    CONTINUE
      DO 30 I = 1,NROW
      DO 30 J = 1,NCOL
30    B(I,J) = A(I,J)
      DO 50 K = 2,N
      CALL MATMY(A,NROW,NCOL,B,NROW,NCOL,C)
      DO 40 I = 1,NROW
      DO 40 J = 1,NCOL
40    A(I,J) = C(I,J)
50    CONTINUE
      DO 70 I = 1,NROW
      WRITE(NP,60) (A(I,J),J=1,NCOL)
60    FORMAT(5X,11F8.4)
70    CONTINUE
      CALL EXIT
      END
```

PROGRAM 8

```
C*****APPLICATIONS OF MATRIX METHODS - PROGRAM 8
      DIMENSION A(20,20),T(20,20),C(20,20), IHEAD(40)
      NP = 5
      NR = 2
C*****READ HEADING AND IDENTIFICATION OF INVESTIGATOR.
1     READ(NR,10,END=50) IHEAD
10    FORMAT(40A2)
      WRITE(NP,20) IHEAD
20    FORMAT(1H1,40A2//)
C******READ THE DIMENSION OF T.  MUST BE SQUARE
      READ(NR,25) N
25    FORMAT(I2)
C******READ THE ARRAY T
      DO 35 I = 1,N
      READ(NR,30) (T(I,J),J=1,N)
30    FORMAT(10F5.0)
35    CONTINUE
C*****FORM IDENTITY MATRIX
      CALL IDN(A,N)
C****CALCULATE I - T.
      CALL MATSB(A,T,N,N,C)
C****CALCULATE THE INVERSE OF I - T.
      CALL INVER(C,N)
      DO 45 I = 1,N
      WRITE(NP,40) (C(I,J),J=1,N)
40    FORMAT(5X,10F10.4)
45    CONTINUE
      GO TO 1
50    CALL EXIT
      END
```

SUBROUTINE MATMY

```
      SUBROUTINE MATMY(A,NROW,NCOL,B,MROW,MCOL,C)
      DIMENSION A(20,20),B(20,20),C(20,20)
      NP = 5
      IF(NCOL-MROW)20,10,20
20    WRITE(NP,15)
15    FORMAT(5X,'NUMER OF COLUMNS IN FIRST MATRIX MUST EQUAL NUMBER OF
     1ROWS IN SECOND MATRIX')
      CALL EXIT
10    DO 40 I = 1,NROW
      DO 40 J = 1,MCOL
      C(I,J) = 0.0
      DO 40 K = 1,NCOL
40    C(I,J) = C(I,J) + A(I,K)*B(K,J)
      RETURN
      END
```

SUBROUTINE MATSB

```
*ONE WORD INTEGERS
      SUBROUTINE MATSB(A,B,NROW,NCOL,C)
      DIMENSION A(20,20), B(20,20), C(20,20)
      DO 10 I=1,NROW
      DO 10 J=1,NCOL
10    C(I,J) = A(I,J)-B(I,J)
      RETURN
      END
```

SUBROUTINE IDN

```
*ONE WORD INTEGERS
*LIST ALL
      SUBROUTINE IDN (A,N)
      DIMENSION A(20,20)
      DO 20 I=1,N
      DO 10 J=1,N
10    A(I,J) = 0.0
20    A(I,I) = 1.0
      RETURN
      END
```

SUBROUTINE INVER

```
*LIST ALL
*ONE WORD INTEGERS
      SUBROUTINE INVER(X,N)
      DIMENSION X(20,20),A(20,40)
      DO 10 I = 1,N
      DO 10 J = 1,N
      A(I,J) = X(I,J)
      M = J + N
      IF(I-J) 20,15,20
15    A(I,M) =1.0
      GO TO 10
20    A(I,M) = 0.0
10    CONTINUE
      DO 55 K = 1,N
      PIVOT = A(K,K)
      IF(PIVOT) 35,30,35
30    WRITE(5,101)
101   FORMAT(///,5X'ZERO PIVOT')
      CALL EXIT
35    A(K,K) = 1.
      IR = K+1
      M = K+N
      DO 40 J = IR,M
40    A(K,J) = A(K,J)/PIVOT
      DO 55 I = 1,N
      IF(I-K) 45,55,45
45    PIVOT = A(I,K)
      A(I,K) = 0.0
      DO 50 J = IR,M
50    A(I,J) = A(I,J) - PIVOT*A(K,J)
55    CONTINUE
      IR = 2*N
      K = N+1
      DO 60 I = 1,N
      DO 60 J = K,IR
      M = J-N
60    X(I,M) = A(I,J)
      RETURN
      END
```

SUBROUTINE KPLOT

```
       SUBROUTINE KPLOT(C,IROW,S,IR,IP)
C*****THIS SUBROUTINE IS THE SAME AS IPLOT EXCEPT THAT IT SETS UP 3 AXES.
       INTEGER PLANE(41,71),ICHR(20)
       REAL C(20,2)
       DATA IBLK,IX/' ','.'/
       DATA ICHR/'A','B','C','D','E','F','G','H','I','J','K','L','M',
      1 'N','O','P','Q','R','S','T'/
C***** S IS A SCALING FACTOR TO BE USED IF COORDINATES ARE OUT OF RANGE
C***** RANGE IS FROM -20 TO +20
C***** SET IR = 0 TO BLANK OUT GRAPH FRAME
C***** SET IR = 1 TO PUT NEW GRAPH IN WITH PREVIOUS ONE.
C***** SET IP = 0 TO SUPPRESS PRINTING OF THE GRAPH
C***** SET IP = 1 TO PRINT THE GRAPH
C***** NP = NUMBER FOR PRINTER.
       NP = 5
       IF(S - 1.) 5,6,5
5      DO 7 I = 1,IROW
        DO 7 J = 1,2
7      C(I,J) = C(I,J) *  S
6      IF(IR)15,8,15
8      DO 10 I = 1,41
       DO 10 J = 1,71
10     PLANE(I,J) = IBLK
       DO 20 J = 36,71,2
20     PLANE(21,J) = IX
       J = 34
       DO 25 I = 1,10
       PLANE(I+21,J) = IX
25     J = J-2
       DO 30 I = 1,20
30     PLANE(I,36)= IX
15     DO 40 K = 1,IROW
       J = 36 + (5./3. * C(K,1) + .5)
       IF(J) 40,40,35
35     IF(J-71)36,36,40
36     I = 21 - (C(K,2) - .5)
       IF(I) 40,40,37
37     IF(I-41)39,39,40
39     PLANE(I,J) = ICHR(K)
40     CONTINUE
C***** RESTORE MATRIX C IF IT HAS BEEN SCALED.
       IF(S-1.) 42,43,42
42     SI = 1./S
       DO 44 I = 1,IROW
       DO 44 J = 1,2
44     C(I,J) = C(I,J) *SI
43     IF(IP) 45,65,45
45     DO 50 K = 1,41
50     WRITE(NP,60) (PLANE(K,J),J =1,71)
60     FORMAT(5X,71A1)
65     RETURN
       END
```

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STUDENT FORM 1

Request for Help

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name______________________________ Unit No.________

Page________
○ Upper
○ Middle
○ Lower

OR

Section__________
Paragraph________

OR

Model Exam Problem No.______
Text Problem No.______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature__________________________

Please use reverse if necessary.

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STUDENT FORM 2

Unit Questionnaire

Name________________________________ Unit No.______________ Date______________

Institution______________________________ Course No.__________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain____

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain____

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

UMAP

**MODULES AND
MONOGRAPHS IN
UNDERGRADUATE
MATHEMATICS
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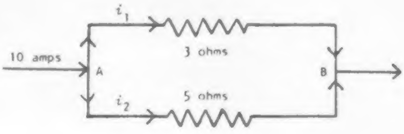
MODULE 108 & 112

Electrical Circuits *(U108)* and Applications of Matrix Methods: Analysis of Linear Circuits *(U112)*

by Sister Mary K. Keller



Applications of Matrix Methods

SE 036 477

Intermodular Description Sheet: UMAP Units 108/112

Title: ELECTRICAL CIRCUITS (U108) and APPLICATIONS OF MATRIX METHODS: ANALYSIS OF LINEAR CIRCUITS (U112)

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Computer Sciences Department
Clarke College
Dubuque, Iowa 52001

Review Stage/Date: IV 7/30/80

Classification: APPL MATRIX/ELEC

Prerequisite Skills:

1. Familiarity with solving systems of equations by matrix methods.

Output Skills:

1. Be able to construct a system of equations representing an electrical circuit using three laws of circuits.
2. Use a computer program to test a system of equations for consistency.
3. Use a computer program to find a unique solution for a system of equations if one exists.
4. Recognize that a system of equations may be over-determined and still have a unique solution.

Related Units:

Food Service Management *(Unit 105)* and Applications of Matrix Methods: Food Service and Dietary Requirements *(Unit 109)*
Computer Graphics *(Unit 106)* and Applications of Matrix Methods: Three Dimensional Computer Graphics and Projections *(Unit 110)*
Markov Chains *(Unit 107)* and Applications of Matrix Methods: Fixed Point and Absorbing Markov Chains *(Unit 111)*

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ELECTRICAL CIRCUITS (U108)

AND

APPLICATIONS OF MATRIX METHODS: ANALYSIS OF LINEAR CIRCUITS (U112)

by

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APPENDIX A 25

1. ELECTRICAL CIRCUITS (U108)

1.1 Introduction

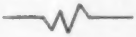
You may be aware that matrix methods play an important part in solving systems of linear equations. We will examine a few aspects of this problem which are treated more completely elsewhere. We turn our attention to the way in which a system of equations might arise from a simple problem in physics dealing with an electrical circuit.

Most people have a general notion of what is meant by an electrical current flowing in a wire. The flow of electrons in a wire is somewhat like the flow of water in a pipe. To produce a flow of current, some source of power is needed, such as a battery. There is also a part of a circuit which consumes power. This is a *resistance*. Current is measured in *amperes*, the source of power in *volts* and resistance in *ohms*.

1.2 Laws for Electrical Circuits

In studying circuits we will use three laws. We first state the laws, then show how they are applied.

1. The sum of all currents flowing to a point equals the sum of the currents flowing away from the point.
2. The algebraic sum of the voltage drops around any loop of a circuit is zero.
3. The voltage drop between two points of a circuit algebraically equals the product of the current and the resistance between the points.

To illustrate these laws we use the circuit shown in Figure 1. The symbol  represents a resistance.

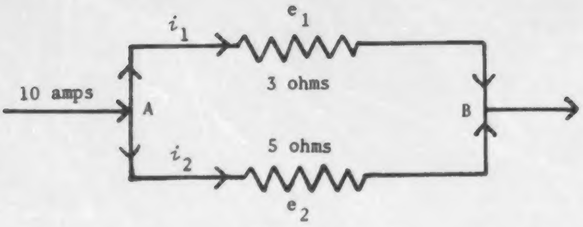


Figure 1.

Ten amperes of current are flowing into point A; therefore 10 amperes must be flowing away from point A. If i_1 and i_2 represent the currents in the upper and lower branches of the circuit, respectively, then

$$i_1 + i_2 = 10. \tag{1.0}$$

Across the resistor in the top branch there is a voltage drop, which we will label e_1. The third law tells us how to calculate this voltage drop from the values of the resistance and the current:

$$e_1 = 3i_1.$$

Similarly, in the bottom branch

$$e_2 = 5i_2.$$

Finally, we note the opposite directions of the current at point B and apply the second law, to obtain the equation

$$3i_1 - 5i_2 = 0. \tag{2.0}$$

1.3 Solving Linear Systems Using an Inverse Matrix

To find i_1 and i_2 in the circuit shown in Figure 1, we now have the system of equations

(1)
$$i_1 + i_2 = 10 \tag{1.0}$$
$$3i_1 - 5i_2 = 0 \tag{2.0}$$

which can be solved easily by substitution. The solution can also be expressed in terms of matrices. The matrix of coefficients of the system in (1) is

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -5 \end{bmatrix}.$$

If we write the unknown values as the column vector

$$X = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix},$$

and the constants on the right hand side of the equations as

$$B = \begin{bmatrix} 10 \\ 0 \end{bmatrix},$$

then the system of equations can be written as

$$AX = B. \tag{2}$$

One way to solve for X is to multiply both sides of Equation (2) by the inverse matrix A^{-1}, to obtain

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B. \end{aligned}$$

The matrix I in this calculation is the identity matrix. Since the inverse of A in our particular case is

$$A^{-1} = \begin{bmatrix} \frac{5}{8} & \frac{1}{8} \\ \frac{3}{8} & -\frac{1}{8} \end{bmatrix},$$

we have

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = X = A^{-1}B = \begin{bmatrix} \frac{5}{8} & \frac{1}{8} \\ \frac{3}{8} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{25}{4} \\ \frac{15}{4} \end{bmatrix}.$$

From this we conclude that

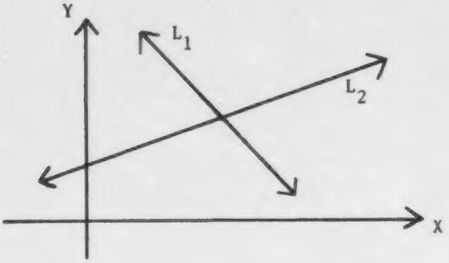
$$i_1 = \frac{25}{4} \text{ amps and } i_2 = \frac{15}{4} \text{ amps.}$$

1.4 Consistent and Inconsistent Systems of Linear Equations

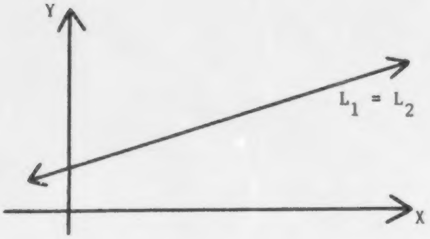
A system of linear equations is said to be *consistent* if there is at least one solution for the system. An *inconsistent* system has no solution. To illustrate this geometrically for a system of two linear equations in two variables, we may represent each equation by a straight line, as in Figure 2.

1.5 Existence Theorems

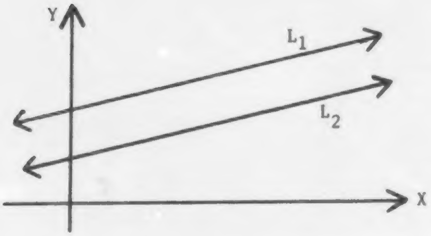
There are theorems about systems of linear equations, called *existence theorems*, that allow us to determine whether a system has no solution, a unique solution, or an infinite number of solutions. As you continue your study of linear algebra, you will learn about these theorems and their proofs. Program 5 in Appendix A applies these theorems to systems of linear equations. You can use this program to investigate the nature of the systems with which you work even though you have not studied the theory.



A. Consistent, one point in common, unique solution.



B. Consistent, all points in common, infinite number of solutions (L_1 and L_2 are two names for the same line).



C. Inconsistent, no points in common, no solution (L_1 and L_2 are parallel).

Figure 2.

1.6 An Example

Let us apply the three laws for electrical circuits to the following circuit.

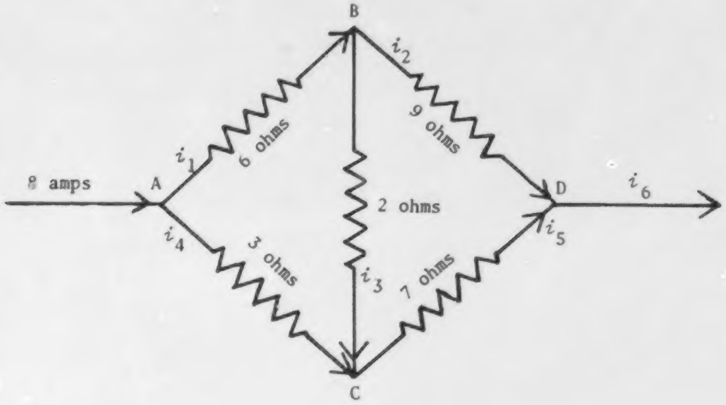

B
i_2
6 ohms
9 ohms
i_1
8 amps
A
2 ohms
D
i_6
i_4
i_5
3 ohms
i_3
7 ohms
C

Figure 3.

Using Law 1 we have the following:

For points: A $i_1 + i_4 = 8$

B $i_2 + i_3 = i_1$

C $i_3 + i_4 = i_5$

D $i_2 + i_5 = i_6$

Using Law 2 we have:

For loops:

ABC $6i_1 + 2i_3 - 3i_4 = 0$

BCD $9i_2 - 7i_5 - 2i_3 = 0$

ABCD $6i_1 + 9i_2 - 7i_5 - 3i_4 = 0$

We can arrange these equations in the form AX = B.

Since we have five variables and seven equations the system seems to be *over-determined*. A system of equations is said to be over-determined when the number of equations is larger than the number of variables involved in the equations. It is possible to make substitutions which eliminate two of the equations. However, it is not necessary to do this. In many practical problems, systems of equations which are derived from physical situations may consist of 50 or even 100 or more equations. One of the benefits of matrix theory is that we can use it to find out whether an over-determined system is consistent, and whether or not the system has a unique solution.

1.7 Experiment I

Consider the system

$$\begin{aligned} 2x + y - z &= 5 \\ x - 4y &= 3 \\ 5x - 2y - 2z &= 13 \\ 6x - 5y - 3z &= 18 \end{aligned}$$

Is the system consistent? Is there a unique solution?

1.8 Model Exam for Unit 108

1. A system of equations which has more equations than unknown variables is called ____.

2. If a system of equations has two equations and two unknowns, the system is ________________________ if the graphs of the equations intersect, the system is ________________________ if the graphs are identical, and the system is ____________ ____________ if the graphs do not intersect.

3. Consistent systems may or may not have ____________________ solutions.

4. Find the values of i_1, i_2, and i_3 for the electrical circuit on the following page.

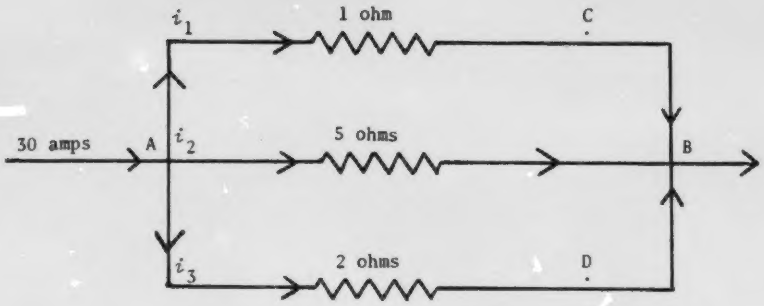


Figure for Model Exam Problem 4.

5. Use Program 5 to investigate the nature of the solution, if it exists, for the following system:

$$\begin{aligned} 2x - y + 3z + w &= 1 \\ x + 4y - z + 2w &= 4 \\ 5x + 2y + 5z + 4w &= 0 \end{aligned}$$

Note that this is an "under-determined" system in the sense that it has more unknowns than equations, but it is still possible to investigate it with our program.

2. APPLICATIONS OF MATRIX METHODS:
ANALYSIS OF LINEAR CIRCUITS (U112)

2.1 Introduction

In Unit 108 on electrical circuits, we considered how a system of linear equations could be used to represent some of the relationships in an electrical circuit. Such a system of equations is useful in linear circuit analysis.* We then explored the use of matrix methods to solve systems of equations, which, in some cases, were over-determined. We also referred to the fact that systems of equations can be consistent or inconsistent, and can have a unique solution, an infinite number of solutions, or no solution. We now explore a simple method to solve such systems.

2.2 Elementary Row Operations

We are interested in the following three types of elementary row operations which may be performed on a matrix:

1. the interchange of any two distinct rows;
2. the multiplication of any row by a nonzero scalar;
3. the addition of a scalar multiple of one row of a matrix to some other row of the same matrix.

If you think of ordinary linear equations, these are the usual ways in which you manipulate them.

---

*A linear circuit for a direct current is one containing elements that obey Ohm's Law, such as metallic conductors. Ohm's Law was given as the third law of electrical circuits on page 1. There are many devices in electronics that do not obey Ohm's Law. They are called "nonlinear."

2.3 Exercises Using Elementary Row Operations

1. Which elementary row operation transforms

$$\begin{bmatrix} 1 & 8 \\ 2 & -1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 8 \\ 0 & -17 \end{bmatrix} \quad ?$$

2. Transform

$$\begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if possible.

2.4 Row Equivalence

A matrix is said to be *row equivalent* to another matrix if the first matrix can be transformed into the second by a sequence of elementary row operations. For example, in the exercises in Section 2.3 the matrix

$$\begin{bmatrix} 1 & 8 \\ 2 & -1 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 8 \\ 0 & -17 \end{bmatrix} \text{ and the matrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Why we are interested in row equivalence will be evident in the discussion which follows. We will see that a transformation of a matrix of coefficients of a system of linear equations which leads to a particular row equivalent matrix is a means of obtaining solutions for the system of equations, if they exist, or in determining that the system is inconsistent.

2.5 Row Echelon Matrices

If you look up the term *echelon* in a dictionary, you will find that it refers to a formation, often used for airplanes or ships, in which there is a lead plane or vessel with the others arranged in step-like fashion slightly to the right or left and to the rear. We use the term echelon here to refer to matrices of a form that, in a way, suggests the meaning of the term as just given. More precisely, we define a matrix to be in *row echelon* form if it has the following properties:

1. in any row of the matrix the first nonzero element at the left must be a 1 unless the row consists of all zeros;
2. rows of all zeros should follow nonzero rows;
3. the column containing the leading (i.e., the leftmost) 1 has zeros elsewhere in the column;
4. the leading 1 of any nonzero row must appear to the left of the leading 1 of the nonzero row that follows it.

The following examples should help to clarify this idea:

1. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (echelon form)
2. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (not in echelon form)
3. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (echelon form)
4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ (not in echelon form).

2.6 Using Row Echelon Form to Solve Systems of Equations

For a given system of linear equations, the coefficients of the variables and the constant terms can be represented as two matrices. For example:

$$\begin{aligned} 2x - 3y &= 6 \\ 5x + y &= -3 \end{aligned}$$

let $V = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

If we write the matrices V and C as one matrix, by writing the constants as a new column on the right:

$$\left[V \mid C \right]$$

and transform this augmented matrix to echelon form, we can find the solution of the original system of linear equations represented by V and C, if the solution exists. Further, this method will expose inconsistent systems, and systems with many solutions. We will demonstrate this with an example. For the above system

$$\left[V \mid C \right] = \left[\begin{array}{cc|c} 2 & -3 & 6 \\ 5 & 1 & -3 \end{array}\right];$$

multiply row 1 by $\frac{1}{2}$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 3 \\ 5 & 1 & -3 \end{array}\right];$$

multiply row 1 by -5 and add to row 2

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 3 \\ 0 & \frac{17}{2} & -18 \end{array}\right];$$

multiply row 2 by $\frac{2}{17}$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 3 \\ 0 & 1 & -\frac{36}{17} \end{array}\right],$$

multiply row 2 by $\frac{3}{2}$ and add to row 1 to achieve row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & -\frac{3}{17} \\ 0 & 1 & -\frac{36}{17} \end{array}\right].$$

The solution for this system can be read from the row echelon form as

$$x = -\frac{3}{17},$$

and

$$y = -\frac{36}{17}.$$

From this problem we can see that transforming a matrix of coefficients to row echelon form, and at the same time applying the row operations to the augmented matrix of coefficients and constants can produce the solution to the system, if it exists.

2.7 Examples and Exercises

Use the method of transforming the augmented matrix to row echelon form to solve each of these systems, if possible. State whether or not the system is inconsistent. The first example shows how this can be detected.

a. $$\begin{aligned} x - y - 2z &= 3 \\ 2x + 3y + z &= 1 \\ 4x + y - 3z &= 5 \end{aligned}$$

The answer is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right]$.

This system is inconsistent. Look at row 3. There is no value of x, y, or z such that $0 = -2$.

b.
$$\begin{aligned} x + 2y - z &= 4 \\ -x - y + 2z &= -5 \\ 4x + 11y - z &= 14 \end{aligned}$$

c.
$$\begin{aligned} 3x + 2y + z &= 5 \\ 2x + 5y + 4z &= 8 \\ x + 4y + 6z &= 4 \end{aligned}$$

d.
$$\begin{aligned} x + y + z &= 3 \\ x - y - z &= -4 \\ 3x + y + z &= 2 \end{aligned}$$

Answer: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$.

From the first row $x = -\frac{1}{2}$, and from the second row $y + z = \frac{7}{2}$. The third row indicates that $0 = 0$. This system is consistent, but there is no unique solution. We can, however, find a particular solution if we assign some value to y or z. For example, let $z = k$, then $y = \frac{7}{2} - k$. For $k = 1$, $y = \frac{5}{2}$.

In this case, the system is consistent but has an infinite number of solutions. We refer to such systems as *consistent systems with parameters*. In the example above k is a *parameter*. A system may have more than one parameter.

e. $$\begin{aligned} x - y &= 1 \\ 2x + y &= 7 \\ 3x - y &= 3 \end{aligned}$$ Does this system have a parameter? Why?

f. $$\begin{aligned} x - 2y &= -2 \\ x - y &= 2 \\ 2x + y &= 5 \end{aligned}$$

g. $$\begin{aligned} x + 4z &= 5 \\ -2x + y + z &= 5 \\ 2x + y + 17z &= 13 \end{aligned}$$

h. $$\begin{aligned} x + 3y - z &= 4 \\ 2x - y + 2z &= 3 \\ 3x + 2y + z &= 7 \end{aligned}$$

2.8 Electric Circuits Revisited

In Unit 108 we used matrices to analyze an electric circuit. The program is quite adequate for even a large system. However, at the time, we started by formulating equations based on the laws of circuits and then constructed a matrix from this system. If a system is more complicated, this may be a nontrival task.

It is possible to formulate the electric circuit problem in terms of matrices from the beginning without writing the equations. This will be illustrated in the following example.

2.9 An Example of an Electrical Circuit

Before we start the example, it is necessary to discuss, briefly, network branches and sign conventions. Each branch of a network in an electrical circuit can be represented as shown in Figure 1.

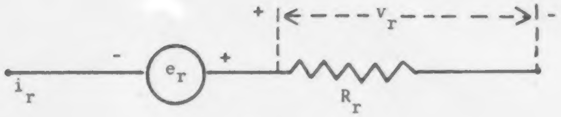


Figure 1.

The voltage drop across the branch is given by $v_r - e_r$, where e_r is an *electromotive force* in series with v_r. Figure 1 shows the sign conventions used for a branch.

Figure 2 shows an example of an electrical circuit using the conventions illustrated in Figure 1.

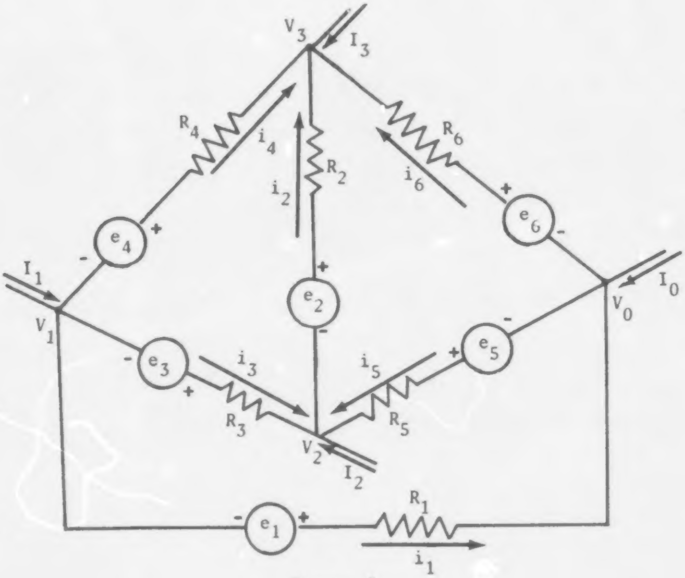

V_3
I_3
R_4
i_4
R_2
i_2
R_6
i_6
+
e_6
-
e_4
+
-
+
e_2
-
I_1
V_1
I_0
V_0
-
e_3
+
i_3
R_3
i_5
+
e_5
-
R_5
V_2
I_2
R_1
-
e_1
+
i_1

Figure 2.

Each branch is connected to the rest of the network at precisely two points, or *nodes*. We number these nodes in Figure 2, arbitrarily, from 0 to 3. We are interested in only the voltage drop across a branch, that is, differences such as $V_2 - V_3$ which is equal to $v_2 - e_2$, so we can set the voltage at one arbitrarily selected node equal to 0. We choose $V_0 = 0$.

From the laws previously stated in Unit 108, the net current at each node must be zero, and we can write the equations for this circuit as

$$
\begin{aligned}
I_0 &= -i_1 + i_5 + i_6 \\
I_1 &= i_1 + i_3 + i_4 \\
I_2 &= i_2 - i_3 - i_5 \\
I_3 &= i_2 - i_4 - i_6
\end{aligned}
$$

However, instead of writing these equations, especially if the system is large and complicated, we can construct the matrix S which preserves the signs of the system. This matrix can be constructed directly from the diagram without the necessity of writing the equations.

$$
S = \begin{array}{c|cccccc|}
I_0 & i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\
I_1 & 1 & 0 & 1 & 1 & 0 & 0 \\
I_2 & 0 & 1 & -1 & 0 & -1 & 0 \\
I_3 & 0 & -1 & 0 & -1 & 0 & -1
\end{array}
$$

Verify that the equations could be obtained from the product

$$J = Si$$

where

$$i = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}.$$

From the diagram in Figure 2 we can construct the following matrices.

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad J = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}.$$

Since the law of electric circuits

$$v = Ri$$

holds for the voltage drop across each resistor in the circuit shown in Figure 2, we have

$$v_n = R_n i_n;\ n = 1, 2, \ldots, 6,$$

or, in matrix notation,

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} .$$

Satisfy yourself that this equation holds.

All our information is now organized in the matrices S, i, V, J, v, e, and R.

From the laws of electrical circuits the following relationships are true:

$v = Ri$ (2.1)

$S^TV = v - e$ (2.2) where ^{T} indicates the transpose

$J = Si$ (2.3)

where all the variables in Equations 2.1, 2.2, and 2.3 represent the matrices constructed above.

2.10 Experiment I

Using Figure 3 construct the matrices S and R for this system. Let J = 0 and solve for V.

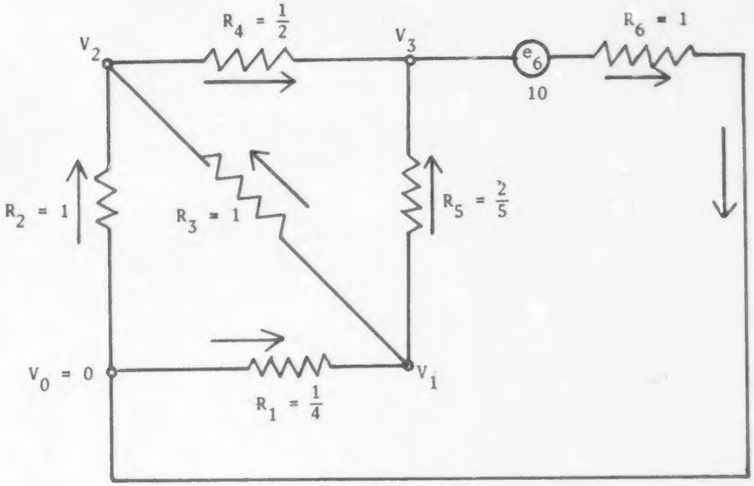

$R_4 = \frac{1}{2}$
V_2
V_3
e_6
10
$R_6 = 1$
$R_2 = 1$
$R_3 = 1$
$R_5 = \frac{2}{5}$
$V_0 = 0$
V_1
$R_1 = \frac{1}{4}$

Figure 3.

The values for the e's not shown are all zero.

2.11 Model Exam for Unit 112

1. Transform this matrix to row echelon form:

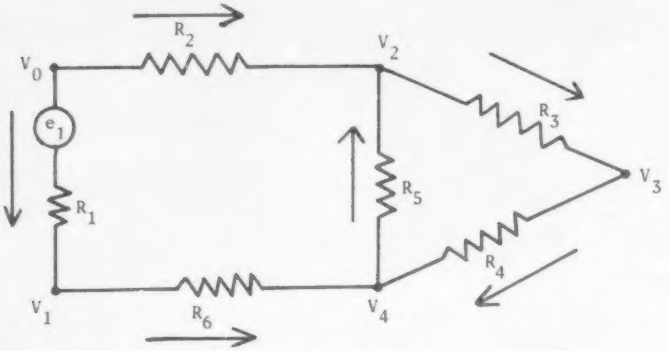
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -3 & 2 \end{bmatrix}.$$

2. Solve this system of linear equations, if possible.

$$\begin{aligned} x - y + 3z &= 2 \\ 2x + 2y + 2z &= 2 \\ 3x + y + 5z &= 4 \\ x + 5y - 3z &= -1. \end{aligned}$$

3. Is the system in Problem 2 consistent? Does it have parameters?

4. Create the necessary matrices for the analysis of the electrical circuit in Figure 4 such that the formula $SR^{-1}S^{T}V = J - SR^{-1}e$ can be used. Do not solve the system.



R_2
V_0
V_2
e_1
R_3
R_5
V_3
R_1
R_4
V_1
R_6
V_4

Figure 4.

3. ANSWERS TO MODEL EXAM (UNIT 108)

1. Over-determined.

2. Consistent, consistent, inconsistent.

3. Unique.

4. N = 4, M = 3.

```
AUGMENTED MATRIX

          1.00      1.00      1.00     30.00
          1.00     -5.00      0.00      0.00
          0.00      5.00     -2.00      0.00
          1.00      0.00     -2.00      0.00

UNIQUE SOLUTION VECTOR X IS

          X(1) = 17.64
          X(2) =  3.52
          X(3) =  8.82
```

5. N = 3, M = 4.

```
AUGMENTED MATRIX

          2.00     -1.00      3.00      1.00      1.00
          1.00      4.00     -1.00      2.00      4.00
          5.00      2.00      5.00      4.00      0.00

EQUATIONS ARE INCONSISTENT.
```

4. ANSWERS TO SOME EXERCISES FROM UNIT 112

Exercises from Section 2.3:

The multiplication of row 1 by a -2 and addition of the results to row 2 will transform

$$\begin{bmatrix} 1 & 8 \\ 2 & -1 \end{bmatrix} \text{ to } \begin{bmatrix} 1 & 8 \\ 0 & -17 \end{bmatrix}.$$

To transform $\begin{bmatrix} 2 & -1 \\ 1 & 7 \end{bmatrix}$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

multiply row 1 by $\frac{1}{2}$ $\begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 7 \end{bmatrix}$,

multiply row 1 by $\frac{1}{2}$ and add the results to row 2 $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 7\frac{1}{2} \end{bmatrix}$,

multiply row 2 by $\frac{1}{7\frac{1}{2}}$ $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$,

multiply row 2 by $\frac{1}{2}$ and add the results to row 1 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5. ANSWERS TO MODEL EXAM (UNIT 112)

1. The row echelon form is:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

2-3. Using Program 9 we get the following result:

```
ROW ECHELON FORM

        1.0000    0.0000     2.0000     1.5000
        0.0000    1.0000    -1.0000    -0.5000
        0.0000    0.0000     0.0000     0.0000
        0.0000    0.0000     0.0000     0.0000
```

From the row echelon form we can see that the system is consistent, but that it has parameters.

Using Program 5 we can find particular solutions. You might want to discuss this result in class.

```
PARTICULAR SOLUTION VECTOR IS

X(1) = 0.00
X(2) = 0.24
X(3) = 0.75

LINEAR INDEPENDENT VECTORS ARE

U(1) to U(1)

      -1.00
       0.50
       0.49
```

4.
$$i = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} \quad V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} \quad R = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $J = Si$.

APPENDIX A

PROGRAM 5

```
      INTEGER HEAD (40)
      DIMENSION AA(10,10),BB(10),X(10),U(10,10)
C****READ ONE LINE HEADING WITH STUDENT NAME
C**** A /* TERMINATES THE RUN
100   READ(2,20,END=50) HEAD
20    FORMAT(40A2)
      WRITE(5,21) HEAD
21    FORMAT(1H1,5X,40A2//)
C****READ THE DIMENSIONS OF THE SYSTEM
      READ(2,1C) N,M
10    FORMAT(2I5)
      WRITE(5,33) N,M
33    FORMAT(5X,'N = ',I3,5X,'M = ',I3//)
      WRITE(5,34)
34    FORMAT(5X'AUGMENTED MATRIX'/)
      DO 1 I = 1,N
C****READ THE COEFFICIENTS AND CONSTANTS
C****THESE ARE PUNCHED IN FIVE COLUMNS EACH WITH A DECIMAL POINT.
C****CHANGE THIS PROGRAM IF THIS FORMAT IS NOT SATISFACTORY
C****THIS IS A SHORT CALLING PROGRAM AND CAN BE ADJUSTED EASILY
      READ(2,2) (AA(I,J),J=1,M),BB(I)
2     FORMAT(11F5.0)
      WRITE(5,35) (AA(I,J),J= 1,M),BB(I)
35    FORMAT(5X,11F10.2)
1     CONTINUE
      CALL SOLEQ(AA,N,M,BB,X,K,U)
      IF(K)100,41,42
41    WRITE(5,5)
5     FORMAT(//5X,'UNIQUE SOLUTION VECTOR X IS'/)
      GO TO 36
42    WRITE(5,43)
43    FORMAT(//5X,'PARTICULAR SOLUTION VECTOR IS'/)
36    DO 32 I = 1,M
      WRITE(5,31) I, X(I)
31    FORMAT(5X,'(X',I3,') = ',F8.2)
32    CONTINUE
      IF(K)100,100,40
40    WRITE(5,6)
6     FORMAT(//5X'LINEARLY INDEPENDENT VECTORS ARE'/)
      WRITE(5,7) K
7     FORMAT(5X'U(1)  TO U('I2,')'/)
      DO 4 I = 1,M
4     WRITE(5,3) (U(I,J),J=1,K)
3     FORMAT(5X,10F8.2)
      GO TO 100
50    CALL EXIT
      END
```

PROGRAM 9

```
C***** PROGRAM 9 - APPLICATIONS OF MATRIX METHODS
      DIMENSION AA(20,20),IHEAD(40),A(20,20)
      NR = 2
      NP = 5
5     READ(NR,10,END=60) IHEAD
10    FORMAT(40A2)
      WRITE(NP,20) IHEAD
20    FORMAT(1H1,40A2//)
      WRITE(NP,25)
25    FORMAT(5X,'INPUT DATA',//)
      READ(NR,30) NROW,NCOL
30    FORMAT(2I5)
      WRITE(NP,21) NROW, NCOL
21    FORMAT(5X,'ROWS = ',I3,5X,'COLUMNS = ',I3,//)
      NCOL = NCOL + 1
      DO 40 I = 1,NROW
      READ(NR,35) (AA(I,J),J = 1,NCOL)
35    FORMAT(10F5.0)
      WRITE(NP,45) (AA(I,J)J = 1,NCOL)
40    CONTINUE
      CALL ECHEL(AA,A,NROW,NCOL)
      WRITE(NP,41)
41    FORMAT(//5X,'ROW ECHELON FORM'//)
      DO 50 I = 1,NROW
      WRITE(NP,45) (A(I,J),J=1,NCOL)
45    FORMAT(5X,10F10.4)
50    CONTINUE
      GO TO 5
60    CALL EXIT
      END
```

SUBROUTINE ECHEL

```
*LIST ALL
*ONE WORD INTEGERS
C***** RETURNS ROW ECHELON FORM OF A MATRIX.
      SUBROUTINE ECHEL(A,AK,NROW,NCOL)
      DIMENSION A(20,20), AK(20,20)
      DO 10 I = 1,NROW
      DO 10 J = 1,NCOL
10    AK(I,J) = A(I,J)
      K = 1
      NC = NCOL - 1
C***** RETURN IF COEFFICIENT MATRIX ROWS REMAINING ARE ALL ZEROS
15    DO 20 I = K,NROW
      DO 20 J = 1,NC
      IF(AK(I,J))30,20,30
20    CONTINUE
      RETURN
C***** FIND THE FIRST NONZERO COLUMN ENTRY
30    DO 40 J = K,NC
      DO 40 I = K,NROW
      IF(AK(I,J))50,40,50
40    CONTINUE
50    IC = J
      IR = I
      DO 55 J = 1,NCOL
      C = AK(IR,J)
      AK(IR,J) = AK(K,J)
55    AK(K,J) = C
      X = AK(K,IC)
      DO 60 J = IC,NCOL
      AK(K,J) = AK(K,J)/X
60    CONTINUE
      DO 70 I = 1,NROW
      IF(I-K) 65,70,65
65    D = AK(I,IC)
      DO 67 J = IC,NCOL
      W= AK(K,J) * D
      AK(I,J) = AK(I,J) - W
      IF(ABS(AK(I,J)) - .0001*ABS(W))66,67,67
66    AK(I,J) = 0.0
67    CONTINUE
70    CONTINUE
C***** RETURN IF LAST ROW HAS BEEN PROCESSED.
      IF(K - NROW)80,75,75
75    RETURN
80    K = K + 1
      GO TO 15
      END
```

SUBROUTINE SOLEQ

```
*LIST ALL
** NUMERICAL ANALYSIS SUBROUTINE SOLEQ
      SUBROUTINE SOLEQ(AA,NI,M,BB,X,K,U)
      DIMENSION AA(10,10),BB(10),A(10,11),X(10),ID(10),U(10,10)
      N=NI
      MM=M+1
      DO 200 I=1,N
      A(I,MM)=BB(I)
      DO 200 J=1,M
200   A(I,J)=AA(I,J)
      K=1
      IF(N-M)15,1,1
15    IT=N+1
      N=M
      DO 16 I=IT,M
      DO 16 J=1,MM
16    A(I,J)=0
1     CONTINUE
      DO 21 I= 1,M
21    ID(I)=I
2     CONTINUE
      KK=K+1
      IS=K
      IT=K
      B=ABS(A(K,K))
      DO 3 I=K,N
      DO 3 J=K,M
      IF(ABS(A(I,J))-B)3,3,31
31    IS=I
      IT=J
      B=ABS(A(I,J))
3     CONTINUE
      IF(IS-K)4,4,41
41    DO 42 J=K,MM
      C=A(IS,J)
      A(IS,J)=A(K,J)
42    A(K,J)=C
4     CONTINUE
      IF(IT-K)5,5,51
51    IC=ID(K)
      ID(K)=ID(IT)
      ID(IT)=IC
      DO 52 I=1,N
      C=A(I,IT)
      A(I,IT)=A(I,K)
52    A(I,K)=C
5     CONTINUE
      IF(A(K,K))71,61,71
61    KK=K
      K=K-1
      DO 62 J=KK,M
62    A(J,J)=-1
      GO TO 6
71    IF(K-N)81,72,120
```

SUBROUTINE SOLEQ (Cont.)

```
72     A(N,MM)=A(N,MM)/A(N,N)
       GO TO 7
81     DO 8 J=KK,MM
       A(K,J)=A(K,J)/A(K,K)
       DO 8 I=KK,N
       W=A(I,K)*A(K,J)
       A(I,J)=A(I,J)-W
       IF(ABS(A(I,J))-.0001*ABS(W))82,8,8
82     A(I,J)=0.
8      CONTINUE
       IF(K-M)22,6,120
22     K=KK
       GO TO 2
6      CONTINUE
       DO 73 I=KK,N
       IF(A(I,MM))120,73,120
73     CONTINUE
7      CONTINUE
       K1=K-1
       DO9IS=1,K1
       I=K-IS
       II=I+1
       DO 9IT=II,K
       DO 9J=KK,MM
       A(I,J)=A(I,J)-A(I,IT)*A(IT,J)
9      CONTINUE
       DO10I=1,M
       DO10J=1,M
       IF(ID(J)-I)10,111,10
111    X(I)=A(J,MM)
11     CONTINUE
       IF(K-M)101,10,101
101    DO 102 IS=KK,M
       ISUB=IS-K
102    U(I,ISUB)=A(J,IS)
10     CONTINUE
       K=M-K
       RETURN
120    K = -1
       WRITE(5,1000)
       RETURN
1000   FORMAT(27H EQUATIONS ARE INCONSISTENT)
       END
```

SE 036 477

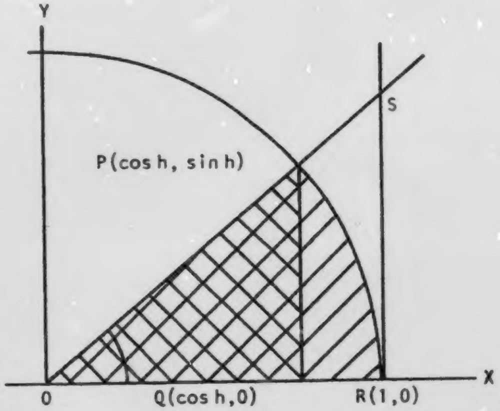
umap

UNITS 158 - 161

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

DERIVATIVES OF SINES AND COSINES

by C. William Stegemoller



DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

edc/umap /55chapel st./newton, mass. 02160

DERIVATIVES OF SINES AND COSINES

by

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Intermodular Description Sheet: *UMAP Units 158-161*

Title: DERIVATIVES OF SINES AND COSINES

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Mathematics Department
Indiana State University
Evansville, Indiana 47712

Review Stage/Date: III 2/2/79

Classification: DERIV TRIG FNCTNS

Suggested Support Material:

Description: This module is introduced by Unit 158, which presents four challenge problems. The three units that follow are designed to provide the skill and understanding to work these problems. In Unit 159 we approximate the derivatives of $y = \sin x$ and $y = \cos x$ at various x-values, using geometric and numerical methods. This leads to conjectures about the derivatives. In Unit 160, the conjectures are validates and applied. Unit 161 then develops formulas for the derivatives of the other trigonometric functions and provides practice in their application.

Prerequisite Skills:

1. Know the definitions of the trigonometric functions.
2. Be familiar with radian measure for angles.
3. Be acquainted with the fundamental trigonometric identities, including the double angle formulas.
4. Be able to draw and to recognize the graphs of simple expressions in which trigonmetric functions appear.
5. Know the chain rule for differentiation and the rules for differentiating sums, products, and quotients.
6. Be able to evaluate simple definite integrals.
7. Know how to calculate the area of a circular sector.
8. Be able to use the rules for calculating limits of sums, products and quotients of functions whose limits are known.

Output Skills:

Unit 158

1. Be able to identify problems that involve calculus applied to trigonometric functions.
2. Be able to describe a problem that involves the calculus of trigonometric functions.

Unit 159

1. Know what units x must be in to make $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ true.
2. Be able to estimate the values of the the derivatives of $y = \sin x$ or $y = \cos x$ for any given x value, where radian measure is used.

Unit 160

1. Know that $\frac{d}{dx}(\sin x) = \cos x$ because $\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim\limits_{h\to 0} \frac{\cos h - 1}{h} = 0$.
2. Know why $\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$.
3. Know that when x is measured in degrees $\frac{d}{dx}(\sin x) = \cos x\left(\frac{\pi}{180}\right)$ and $\frac{d}{dx}(\cos x) = -\sin x\left(\frac{\pi}{180}\right)$.
4. Be able to differentiate and antidifferentiate simple functions expressed in terms of sines and cosines.
5. Be able to solve the challenge problems of Unit 158, referring to the discussion when necessary.

Unit 161

1. Know differentiation formulas for all six trigonometric functions.
2. Given the derivatives of $\sin x$ and $\cos x$, be able to derive the derivatives of the other four trigonometric functions.
3. Be able to differentiate simple expressions involving sums, products and quotients of trigonometric functions.

Other Related Units:

Five Applications of Max-Min Theory from Calculus *(Unit 341)*

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MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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UNIT 158: CHALLENGE PROBLEMS

1. CHALLENGE PROBLEMS 1
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   1.2 Out Fishing 1
   1.3 Putter Gutters 2
   1.4 Average Power 3
   1.5 Pulling a Box 5
2. MODEL EXAM 8
3. ANSWERS TO MODEL EXAM 9

1. CHALLENGE PROBLEMS

1.1 Introduction

You have used calculus to solve problems that would have been either impossible or much more difficult without calculus. In the problems posed here you will find that calculus will lead to the solution, but that calculus must be applied to trigonometric functions.

Read through each problem carefully. Decide which concepts and procedures from calculus are needed to solve each problem. Then, after your study of differentiation and integration applied to trigonometric functions in Unit 159 - Unit 161, you should be able to find the solutions to the problems.

1.2 Out Fishing

Jack Jukes is out fishing on a spring afternoon. First, there is no wind and his cork is perfectly still in the water. Later in the afternoon a wind comes up causing the cork to bob up and down.

From his physics course of the previous semester Jack knows that the vertical position of the cork plotted as a function of time will be a sine curve. The graph of the position of Jack's cork with respect to time is shown in Figure 1.

(Note: If there were no wind, the position of the cork would remain stationary at $y = 0$ as t increased. Also, $t = 0$ is exactly 2:00:00 p.m.) With all this information magically at his disposal Jack asks himself, "What is the position of my cork and how fast is its position changing at ½ second and at 1½ seconds after 2:00:00 p.m.? Also, at what point during the first 2 seconds is my cork falling fastest?"

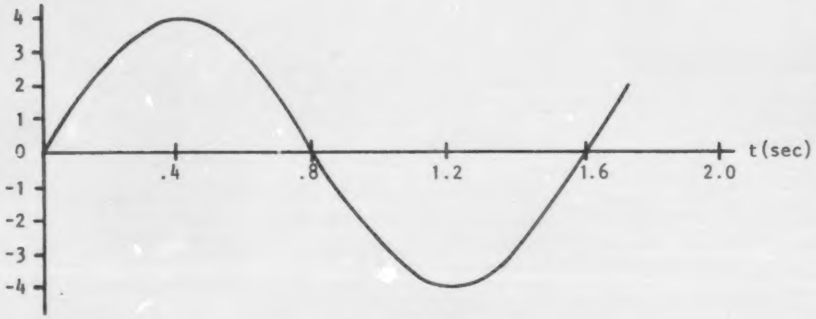


Figure 1. Graph of the position of Jack's cork.

1.3 Putter Gutters

The Putter Gutter Company is planning to make gutters from 14-inch strips of galvanized steel. They are to be designed as shown in Figure 2.

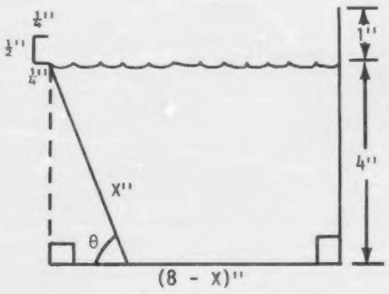


Figure 2. Section of gutter.

As illustrated, one inch on the outside edge will be used for the lip and one inch on the inside edge will be used for securing the gutter to a building. It is also desired that the total length of the side against the building be five inches and that the bottom be perpendicular to the side against the building as shown. The final consideration is to design the gutter such that it will hold the most water possible when filled to a depth of four inches. The questions that need to be answered are, "Where should the bend between the bottom and outer side be?" and "What will the angle that the outer side is to be bent up (angle θ in the illustration) be?"

1.4 Average Power

Electrical power, measured in watts, is the product of the impressed voltage and the resulting current in amperes. We have

$$p(\text{watts}) = v(\text{volts}) \times i(\text{amperes}).$$

When resistance is measured in ohms, we also have

$$i(\text{amperes}) = \frac{v(\text{voltage})}{R(\text{ohms})}.$$

When the voltage for alternating current (AC) is graphed with respect to time, the result is a sine curve. Suppose AC voltage is given by the equation

$$v = 170 \sin \frac{\pi}{12} t,$$

where t is in microseconds.

Also, suppose the resistance in a circuit is 17 ohms, then

$$i = \frac{170 \sin \frac{\pi}{12} t}{17} = 10 \sin \frac{\pi}{12} t.$$

Now the equation for power is

$$p = (170 \sin \frac{\pi}{12} t)(10 \sin \frac{\pi}{12} t),$$

where p is in watts.

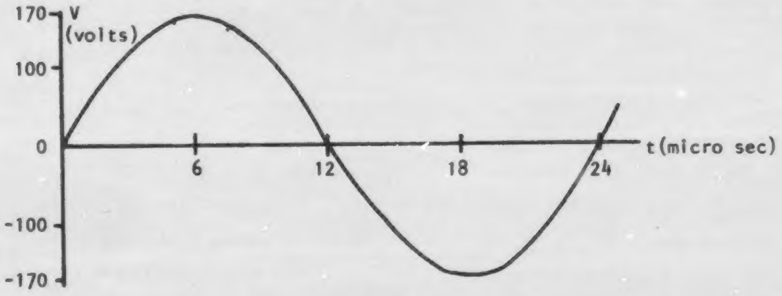


Figure 3. Graph of $v = 170 \sin \frac{\pi}{12} t$.

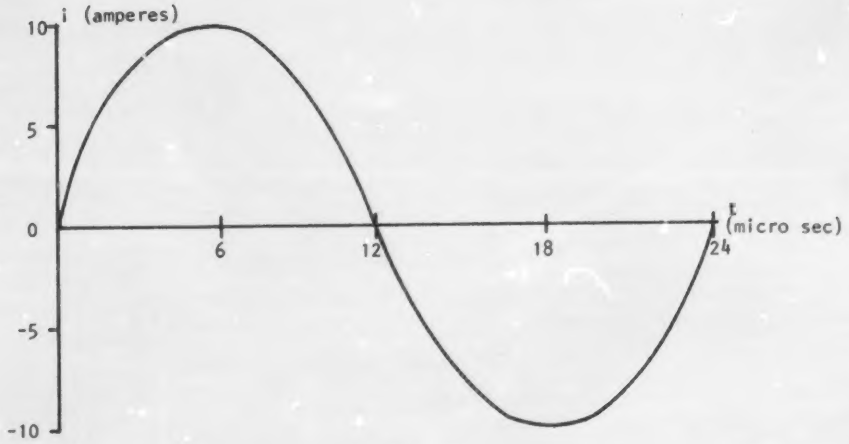


Figure 4. Graph of $i = 10 \sin \frac{\pi}{12} t$.

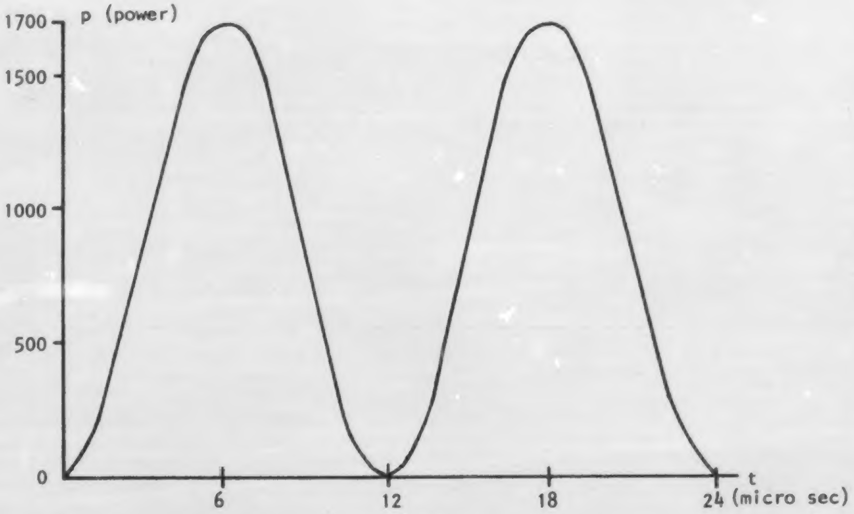


Figure 5. Graph of $p = 1700 \sin^2 \frac{\pi}{12} t$.

The average power P, where p is a periodic function of time t with period T, is defined as follows:

$$\text{(average power)} \qquad P = \frac{1}{T}\int_0^T p\, dt.$$

What is the average power where $p = 1700 \sin^2/12\ t$? What is the geometric interpretation of P?

1.5 Pulling a Box

Jason Baxter and Sam Jones are having an argument concerning pulling a heavy box across a long room. They have a rope tied to the box and Sam says, "We should pull parallel to the floor." Jason says, "It is better to pull at an angle."

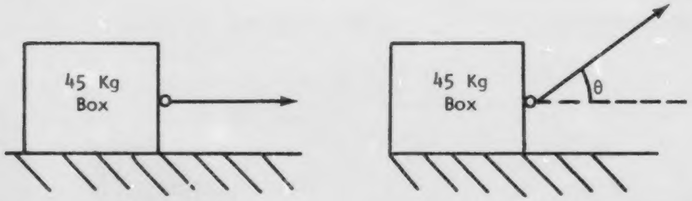


Figure 6a. Sam's Proposal. Figure 6b. Jacob's Proposal.

Figure 6. Proposed pulling angles.

You are called in to settle the argument. You begin by recalling your recent physics course. First, you recall "coefficient of friction." Friction or resistance varies for different surfaces. If it requires a force of magnitude F kg directed paralled to a horizontal surface to pull an object of weight W kg steadily across the surface, then the coefficient of friction K is the ratio of F to W. That is,

$$K = \frac{F}{W}.$$

For example, if a horizontal force of 6 kg will move a box weighing 45 kg steadily across the floor, then the coefficient of friction between the box and the floor is

$$K = \frac{6\text{kg}}{45\ \text{kg}} = .133.$$

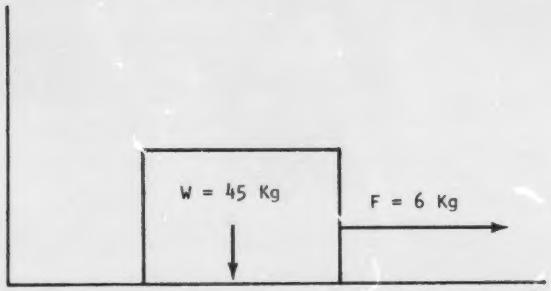


Figure 7. The coefficient of friction is $K = \frac{6}{45} = 0.133$.

In considering the problem you assume the weight is concentrated in a single point and when the force is applied to an angle θ as suggested by Jason, Figure 8 illustrates the situation.

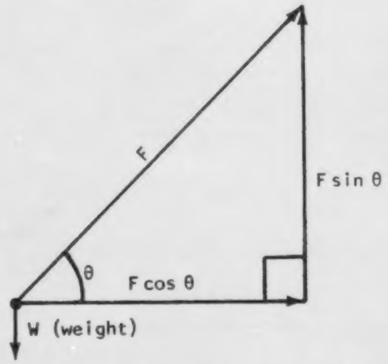


Figure 8. Magnitudes of forces acting on box.

In this case, the upward component of the applied force nullifies part of the downward force of the box, giving $(W - F \sin\theta)$ as our replacement for W in determining the coefficient of friction. Since the magnitude of the applied force parallel to the floor is given by $F\cos\theta$, this is our replacement for F to determine the coefficient of friction. Thus, the coefficient of friction is

$$K = \frac{F \cos\theta}{W - F\sin\theta}.$$

Solving this equation for F we obtain

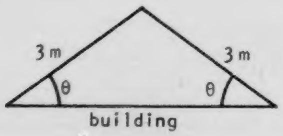
$$F = \frac{KW}{K\sin\theta + \cos\theta}\,\text{kg}.$$

Now, your problem is to find the value of θ that minimizes F, and resolve the argument.

2. MODEL EXAM

Read each of the following problems carefully and decide whether calculus is needed to solve them.

1. The angle of elevation of the top of a television tower from a point 1200 meters away is 0.3 radians. What is the height of the tower?
2. Suppose that the resultant sound from guitar strings vibration has a voltage (v) given by $v = \sin 2t - 2\sin(t + \frac{\pi}{4})$. What is the maximum voltage where $0 \le t \le \frac{\pi}{2}$?
3. Suppose that a 14" (diameter) pizza is cut through the center in such a way that a particular piece forms an angle measuring 120°. What is the area of this piece of pizza?
4. Suppose that owners of a store want to put a triangular sign on top of their building which is over 6 meters long. They want the sign to be an isosceles triangle and have 6 meters of molding to put around the 2 sides that are above the building. What should θ, the measure of the base angles, be to get a triangle of maximum area?



5. Describe a problem that involves the calculus of trigonometric functions.

UNIT 159: FORMULATING CONJECTURES ABOUT THE DERIVATIVES OF $y = \sin x$ AND $y = \cos x$

1. TANGENT METHOD APPLIED TO $y = \sin x$ AND $y = \cos x$ 1
    - 1.1 Tangents to $y = \sin x$ 1
    - 1.2 Graphing the Derivative of $y = \sin x$ 3
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2. NUMERICALLY CALCULATING DERIVATIVES FOR $y = \sin x$ AND $y = \cos x$ 7
    - 2.1 Introduction 7
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    - 2.3 Applying the Procedure 8
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3. MODEL EXAM 16
4. ANSWERS TO MODEL EXAM 17

1. TANGENT METHOD APPLIED TO $y = \sin x$ AND $y = \cos x$

1.1 Tangents to $y = \sin x$

You may be familiar with the so-called Tangent Method for measuring the slope of a line graphed in a coordinate plane.* In this unit, we are going to use the method to measure the slopes of lines that are tangent to the curve $y = \sin x$ at various values of x. This will give us numerical information about the instantaneous rate of change of the function at these values of x.

We recall from trigonometry that $y = \sin x$, where y is the sine of the angle whose radian measure is x, is a periodic function with a period of 2π. It then seems reasonable to consider x values such that $0 \le x \le 2\pi$. Let us try to pick x values at approximately .5 unit intervals, recalling that the x values of 0, $\pi/2$, π, $3\pi/2$, and 2π are of special significance in graphing trigonometric functions. With these considerations we chose the x values that appear in Table 1.

Notice in Figure 1 which is the graph of $y = \sin x$ that each small subdivision represents 0.1 unit and that each large subdivision represents one unit. It is instructive to use a common reference point to compare the slope of the tangent line to see how the slope changes as x increases. Slide your triangle along your stationary ruler (procedure is explained in Appendix 1) to translate from thr tangent to the curve to a parallel line through the point labeled A to compute the value of the tangent for the x values

---

* See Appendix 1.

TABLE I

x	Slope of tangent to y = sin x
0	1
.5	
1	
π/2	
2	
2.5	
π	
3.7	
4.2	
3π/2	
5.2	
5.7	
2π	

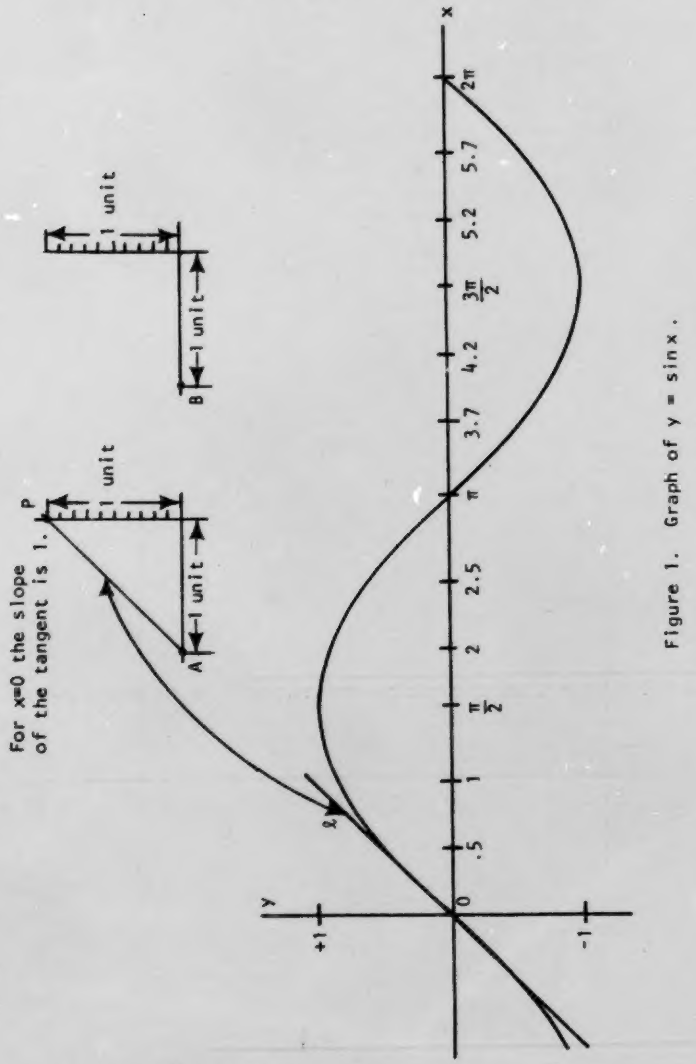


Figure 1. Graph of y = sin x.

of 0, 0.5, 1, π/2, 2, 2,5, and π. Use the same procedure to translate from the tangent line to a parallel line through the point labeled B to compute the value of the tangent for x values of 3.7, 4.2, 3π/2. 5.2, 5.7, and 2π. Record your values to nearest 0.05 unit.

The work for x = 0 is done for you. Line ℓ is tangent to y = sin x at x = 0. Line *m* is parallel to ℓ and contains point A. Line *m* intersects the vertical line, which is 1 unit to the right of point A, at point P. Since P is one unit above the horizontal line through A, the slope of the tangent line is 1/1 or 1.

Note: Although the tangent line ℓ to the curve at x = 0 is drawn in for illustrative purposes, it is advisable not to draw in other tangents. The many lines may cause confusion.

1.2 Graphing Derivative of y = sin x

After completing Table I we will plot the points with coordinates (x,y) where each x is an x value from the table and the y value corresponding to each x is the slope of the tangent to y = sin x. Plot these points on the coordinate system provided in Figure 2. The first point plotted will have coordinates (0,1).

Having plotted the thirteen points using Table I, sketch a smooth curve through (or very close to) all these points. The curve we now have is the graph of the rate of change in y = sin x.

1.3 Making a Guess

Let us hypothesize that this curve is also the graph of a trigonometric function of x. You are now asked, "What is the trigonometric function of form y = f(x) whose graph this curve most closely approximates?" You may wish to refer to any trigonometry book handy to refresh your memory about the graphs of trigonometric functions.

My guess is ______________________.

Before proceeding further refer to page 13 .

Hopefully, your table and graph closely approximated those on page 13 and you guessed the trigonometric function that was given. If you missed some values by 0.2 or more it is advisable to review the procedure in Appendix 1, and try to do the exercise again.

1.4 Tangents to y = cos x

Let us use the Tangent Method again to see if we can guess the function of x which represents the rate of change of y = cos x. Use the procedure you used for y = sin x to complete Table II on page 6. The work is again illustrated for x = 0.

1.5 Graphing the Derivative of y = cos x

With Table II complete we will plot the thirteen points with coordinates (x, y) where each x is an x value from the table and the y value corresponding to each x value is the slope of the line tangent to y = cos x. Use the coordinate axes (Figure 3) on page 5 to plot these points and sketch the curve.

1.6 Guessing Again

The question is again, "What is the trigonometric function of the form y = f(x) whose graph this curve most closely approximates?"

My guess is ______________________.

Refer to page 14 before proceeding.

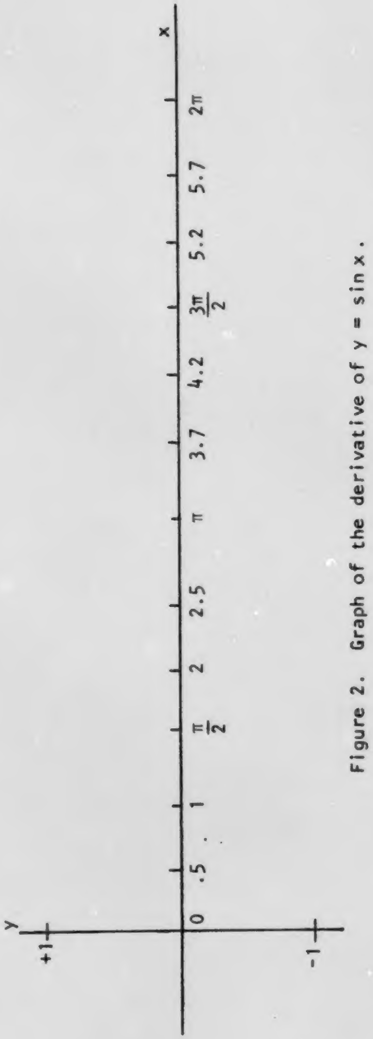


Figure 2. Graph of the derivative of y = sin x.

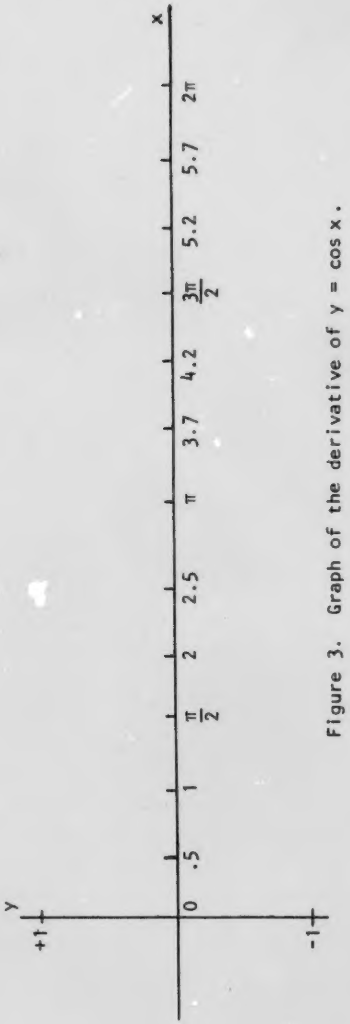


Figure 3. Graph of the derivative of y = cos x.

TABLE II

x	Slope of tangent to y = cos x
0	
.5	
1	
π/2	
2	
2.5	
π	
3.7	
4.2	
3π/2	
5.2	
5.7	
2π	

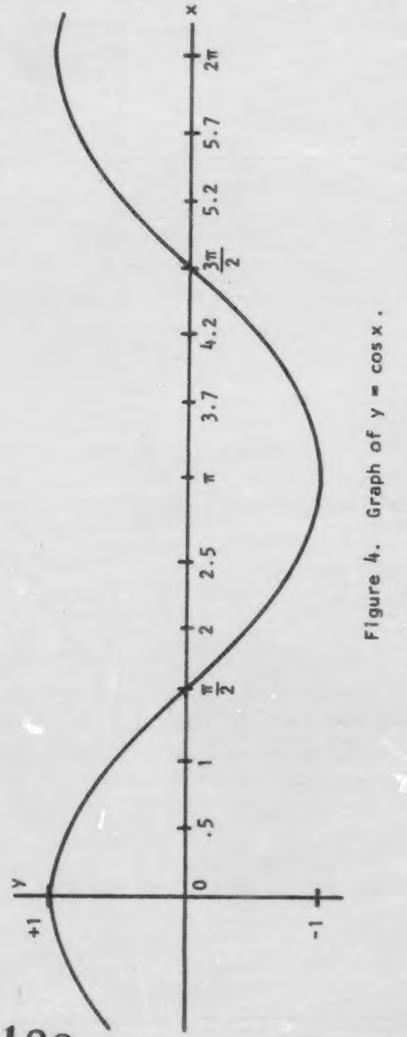


Figure 4. Graph of y = cos x .

2. NUMERICALLY CALCULATING DERIVATIVES FOR y = sin x AND y = cos x

2.1 Introduction

In Section 1 we used the Tangent Method of Appendix 1 to approximate the instantaneous rate of change of y = sin x and y = cos x for various values of x. Then for each function we plotted points P(x,y) where the y value was the instantaneous rate of change of the original function for the given x value. Next, we sketched a smooth curve determined by the points for each of the original functions. Recognizing that this curve represented a function in each case, we guessed an equation for this function. As you know, this new derived function is called the derivative of the original function. Thus, we are led to guess that the derivative of y = sin x is y = cos x (Notation: dy/dx = cos x when y = sin x) and the derivative of y = cos x is y = -sin x (Notation: dy/dx = sin x when y = cos x).

Now that we have formulas for the derivatives that may be correct, let us check further using numerical calculations.[*] For each function, let us numerically calculate the average rate of change over various intervals with a fixed x value (call it x_1) as one end point and numerically approximate the value of the derivative at x_1 by letting the lengths of the intervals approach zero.

2.2 The Procedure Explained

Again, consider y = sin x, where y is the sine of the angle whose radian measure is x, and approximate the value of the derivative at $x = x_1$. Our intervals along the x-axis will have x_1 as one end point and $x_1 + \Delta x$ as the other end point. We want to calculate the ratio of the change in y to the change in x as we move from P_1 to P_2 where P_1 has coordinates (x,sin x_1) and P_2 has coordinates (x_2, y_2) where $x_2 = s_1 + \Delta x$ and $y_2 = \sin(x_1 + \Delta x)$. Now, the change in x is Δx and the change in y is $\Delta y = \sin(x_1 + \Delta x) - \sin x$.

---

[*] See Appendix 2.

We will first choose positive values for Δx (Refer to Figure 5) and pick them so that each successive choice is closer to zero than the preceding one. We will then choose negative values for Δx (refer to Figure 7) again picking them so that each successive choice is closer to zero than the preceding one. By observing Figures 6 and 8 we see that in either case $\Delta y/\Delta x$ should approach the value of the derivative at the point where $x = x_1$. In this way we will get a decimal approximation of the value of the derivative of $y = \sin x$ at $x = x_1$. We will then find the value of cos x_1 and if the approximation is close to the value of cos x_1 we will have further reason to believe that our formula is correct.

2.3 Applying the Procedure

We will now use the procedure just discussed to approximate the value of the derivative of $y = \sin x$ at $x = 0.5$. Figures 5 through 8 illustrate the material just discussed.

We will give 0.8776 as our approximation since we get this value as we approach for both the left and right. We find on our scientific calculator that correct to four decimal places cos .5 = 0.8776. Thus we have further reason to believe that $dy/dx = \cos x$ when $y = \sin x$.

Next, we consider $y = \cos x$ and estimate the derivatives at $x = \pi/3$. In recording values in the

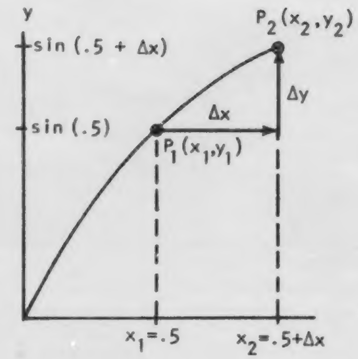


Figure 5. Δx is positive.

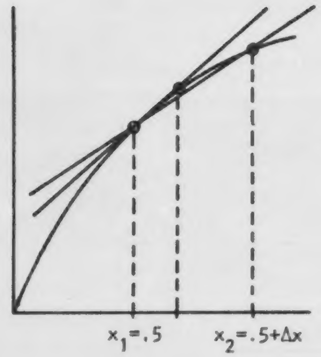


Figure 6. Positive values of Δx approach zero.

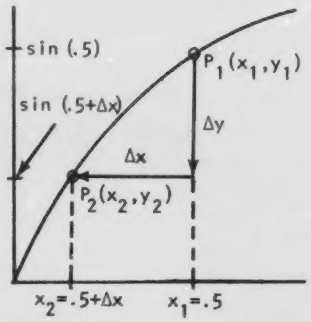


Figure 7. Δx is negative.

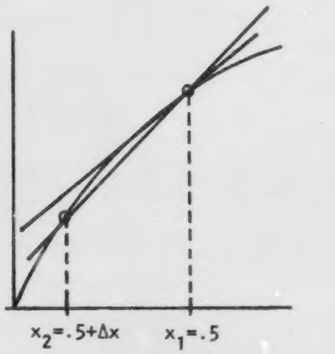


Figure 8. Negative values of Δx approach zero.

Our table to approximate the value of the derivative follows.

TABLE III

Approximating the derivative of $y = \sin x$ at $x = 0.5$

Δx	$\Delta y = \sin(0.5 + \Delta x) - \sin 0.5$	$\Delta y/\Delta x$
0.1	0.08521	0.8521
.01	.008751	.8751
.001	.0008773	.8773
.0001	.00008776	.8776
-.1	-.09001	.9001
-.01	-.008800	.8800
-.001	-.0008778	.8778
-.0001	-.00008776	.8776

y column we list four decimal places plus the number of decimal places in Δx. In the $\Delta y/\Delta x$ column we will record four decimal places.

Exercises

1. You should complete Table IV and the Sentence following the table.

TABLE IV

Approximating the derivative of $y = \cos x$ at $x = \pi/3$

Δx	$\Delta y = \cos(\pi/3 + \Delta x) - \cos(\pi/3)$	$\Delta y/\Delta x$
0.1	-0.8896	-0.8896
.01	-0.008685	-0.8685
.001	______	______
.0001	______	______
-0.1	0.08396	-0.8396
-0.01	.008635	-0.8635
-0.001	______	______
-0.0001	______	______

With our approximation of ______ we find that $-\sin(\pi/3) =$ ____.

2. Use this numerical method to estimate the derivative of $y = \cos x$ and $x = 2$. Compare the result with the value of cos 2.

Δx	$\Delta y = \sin(x + \Delta x) - \sin x$	$\Delta y/\Delta x$
0.1		
.01		
.001		
.0001		
-0.1		
- .01		
- .001		
- .0001		

3. Again, use this numerical method to estimate the derivative of $y = \cos x$ at $x = \pi/6$. Compare the result with the value of $- \sin \pi/6$.

Δx	$\Delta y = \cos(\pi/6 + \Delta x) - \cos \pi/6$	$\Delta y/\Delta x$
0.1		
.01		
.001		
.0001		
-0.1		
- .01		
- .001		
- .0001		

2.4 Using Degree Measure

You may check your results to Exercises 1, 2, and 3 with those on page 12. You now may be reasonably convinced that we have chosen the correct formulas. In all these calculations and in the work in Unit we were evaluating the sine or cosine of an angle given its radian measure.

Let us consider y = sin x where we are evaluating the sine of the angle whose degree measure is x.

Now, we will use the same procedure to approximate the derivative of y = sin x at x = 35. This time we will take the sine of the angle whose degree measure is x. The results appear in Table V.

TABLE V

Trying to approximate the derivative of $y = \sin x$ at $x = 35$ using degree measure

Δx	$\Delta y = \sin(35° + \Delta x) - \sin 35°$	$\Delta y/\Delta x$
5	0.0692	0.0138
3	.0420	.0140
1	.0142	.0142
.1	.00143	.0143
.01	.000143	.0143
-5	-0.0736	0.0147
-3	- .0436	.0145
-1	- .0144	.0144
-0.1	- .00143	.0143
-.01	- .000143	.0143

Thus, our approximation to the derivative of y = sin x and x = 35° is 0.0143. The value of cos 35° = 0.8912.

This is not at all close to what we may have expected from our work in Section 1. In trying to salvage something we recall that all the previous work used radian measure. Maybe we should have stayed with radian measure.

In fact, in Unit 160 we prove that dy/dx = cos x when y = sin x and x is the radian measure of the angle. The problem raised by measuring the angle in degrees has yet to be resolved.

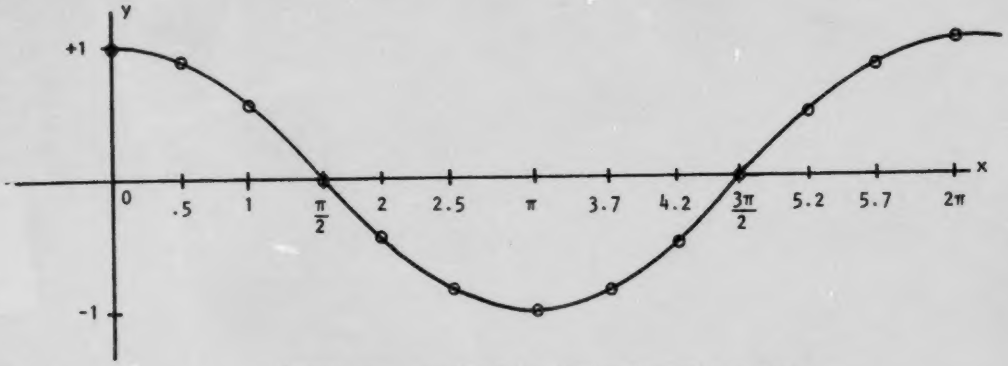


Figure 2a. Graph of the derivative of $y = \sin x$.

Your values in Table I should be close to those listed here in Table IA and your graph in Figure 2 should be similar to Figure 2a above.

The curve graphed in Figure 2a looks like the graph of $y = \cos x$. Was that your guess?

TABLE IA

x	Slope of tangent to $y = \sin x$
0	1
.5	.9
1	.55
π/2	0
2	-.4
2.5	-.8
π	-1
3.7	-.85
4.2	-.5
3π/2	0
5.2	.45
5.7	.8
2π	1

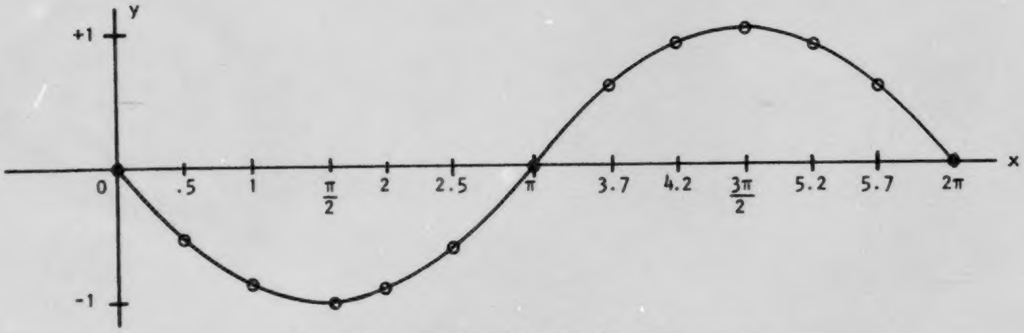


Figure 3a. Graph of derivative of $y = \cos x$.

You should ckeck your entries in Table II with those listed in Table IIA at the right. Your entries should be close to these. Your graph in Figure 3 should look like Figure 3a above.

The curve graphed in Figure 3a looks like the graph of $y = -\sin x$. Did you guess this?

TABLE IIA

x	Slope of tangent to $y = \cos x$
0	0
.5	-.5
1	-.85
π/2	-1
2	-.9
2.5	-.6
π	0
3.7	.5
4.2	.9
3π/2	1
5.2	.9
5.7	.55
2π	0

3. MODEL EXAM

1. Based upon the results of your graphical work and numerical calculations, complete the following statements:

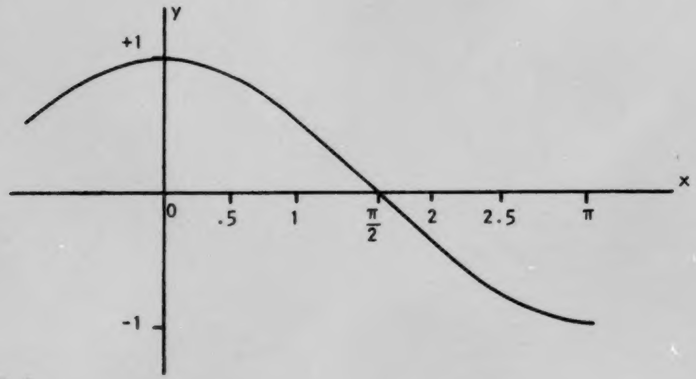
   a. $\frac{d}{dx}$ (sin x) = ____________.

   b. $\frac{d}{dx}$ (cos x) = ____________.

2. Complete the following statements. In determining the derivative of y = sin x graphically, the y value was the sine of the angle whose ____________________ measure was x.

3. Is it important to use a particular unit of measure for angles to get the results that you listed in answering problem 1?

4. From the graph below, determine geometrically the value (to nearest tenth) of the derivative of y = cos x at x_1 = .8 and at x_2 = 2 (radians).



   a. At x = 0.8, the value of $\frac{d}{dx}$ cos x is approximately ____________________.

   b. At x = 2, the value of $\frac{d}{dx}$ cos x is approximately ____________________.

5. Complete the headings, then use a scientific calculator to complete the following table from which you will aprpoximate the value of the derivative of y = sin x at x = 0.4 radians.

Δx	Δy = ____________________	________
.1		
.01		
.001		
.0001		
-.1		
-.01		
-.001		
-.0001		

The value of the derivative y = sin x at x = .4 is approximately ____________________.

UNIT 160: VERIFYING CONJECTURES ABOUT THE DERIVATIVES OF $y = \sin x$ AND $y = \cos x$ AND APPLYING THE RESULTS

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1. PROVING THE FORMULA FOR THE DERIVATIVE OF $y = \sin x$

1.1 Applying the Definition of Derivative

Let us now try to prove our conjecture that $\frac{d}{dx} \sin x = \cos x$ where radian measure of angles is used. We begin by applying the definition of derivative where the Δx used in previous work is replaced by h. We must show that for arbitrary x,

$$\lim_{h\to 0} \frac{\sin(x + h) - \sin x}{h} = \cos x.$$

Now, using $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\lim_{h\to 0} \frac{\sin(x + h) - \sin x}{h}$$

becomes

$$\lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} .$$

Upon collecting the sin x terms and writing as a sum, we have

$$\lim_{h\to 0} \left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x(\sin h)}{h}\right].$$

Applying laws for limits and keeping in mind that x is fixed, we rewrite the previous expression as

$$\sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} .$$

In order for our conjecture to hold up, the first limit must be zero and the second limit must be one.

1.2 Some Numerical Calculations

Before undertaking attempts at proof, let us use a hand calculator to compute values of these expressions for h values close to zero. Complete the following table remembering we are using radian measure for angles.

TABLE I

Considering $\frac{\sin h}{h}$ and $\frac{\cos h - 1}{h}$ for small values of h

h	sin h	$\frac{\sin h}{h}$	cos h	$\frac{\cos h - 1}{h}$
.2				
.1				
.05				
.01				
-.2				
-.1				
-.05				
-.01				

The values just recorded should lead us to believe that

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

and

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

as we had hoped.

1.3 Proof that $\lim_{h \to 0} \frac{\sin h}{h} = 1$

We will now attempt to prove that

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 .$$

Let us consider ∠ROS in standard position in Figure 1.

It is clear that for this acute angle we have Area ΔOPQ < Area Sector OPR < Area ΔORS.

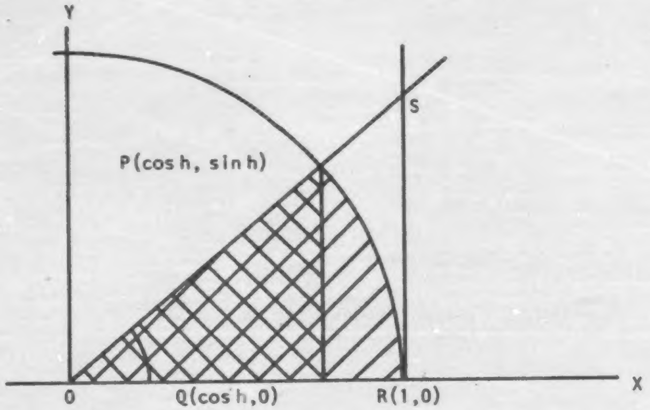

Figure 1. Considering ∠ROS with radian measure h.

Since OP = 1, we find that Q has coordinates (cos h,0) and P has coordinates (cos h, sin h) directly from the definitions of sin h and cos h. Thus Area ΔOPQ = (½)(base)(height) = (½)cos h sin h.

Next, from

$$\frac{\text{Area of Sector}}{\text{Area of Circle}} = \frac{\text{Rdn measure of angle of sector}}{2\pi}$$

we have

$$\frac{\text{Area } O\ P\ R}{\pi \cdot 1^2} = \frac{h}{2} .$$

Now, in order to find Area ΔOSR we need to find RS. Since ΔOPQ ≅ ΔOSR, we have

$$\frac{RS}{\sin h} = \frac{1}{\cos h} .$$

So

$$\begin{aligned} \text{Area } \Delta OSR &= (\tfrac{1}{2})(\text{base})(\text{height}) \\ &= (\tfrac{1}{2})(1)\left(\frac{\sin h}{\cos h}\right). \end{aligned}$$

Substituting in our inequality involving these two triangles and the sector, we have

$$(\tfrac{1}{2})\cos h \sin h < \frac{h}{2} < (\tfrac{1}{2})\frac{\sin h}{\cos h} .$$

Multiplying by $\frac{2}{\sin h}$, which is positive since h is positive, we obtain

$$\cos h < \frac{h}{\sin h} < \frac{1}{\cos h} .$$

Next, we use the fact that where a,b,c,d are all positive, $\frac{a}{b} < \frac{c}{d}$ if $\frac{d}{c} > \frac{b}{a}$. Using this on each half of the compound inequality just obtained we get

$$\frac{1}{\cos h} > \frac{\sin h}{h} > \cos h .$$

Now,

$$\lim_{h \to 0} \frac{1}{\cos h} = \frac{1}{1} = 1$$

and

$$\lim_{h \to 0} \cos h = 1$$

so

$$\lim_{h \to 0} \frac{\sin h}{h}$$

must be 1 since $\frac{\sin h}{h}$ is sandwiched between $\frac{1}{\cos h}$ and cos h.

Now, we need to show that

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

To do this let h = -t, where t > 0. With this substitution we have

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{t \to 0} \frac{\sin (-t)}{-t} .$$

Recalling that sin (-t) = -sin t, we get

$$\lim_{t \to 0} \frac{\sin(-t)}{-t} = \lim_{t \to 0} \frac{-\sin t}{-t} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

and combining this with the proof for $h \to 0$, we have proven that

$$\lim_{h \to 0} \frac{\sin h}{h} = 1,$$

where radian measure of angles is used.

1.4 Proof that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

We will now try to show that

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 1$$

using our last result. Knowing that $-\sin^2 h = \cos^2 h - 1$ we will multiply to obtain an equivalent fraction with $\cos^2 h - 1$ as numerator. Thus.

$$\lim_{h \to 0} \frac{\cos h - 1}{h} \cdot \frac{(\cos h + 1)}{(\cos h + 1)}$$

$$= \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h - 1)} = \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h - 1)}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{-\sin h}{h(\cos h - 1)}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{-\sin h}{h(\cos h - 1)}$$

$$= 0 \cdot \lim_{h \to 0} \frac{-\sin h}{\cos h - 1} = 0$$

1.5 Conclusion

Recall that in Section 1.1 we found that

$$\lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h}$$

$$= \sin x \cdot \lim_{h \to 0} \frac{(\cos h - 1)}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \quad .$$

With the results of Sections 1.3 and 1.4 the previous expression becomes sin x (0) + cos x (1) or cos x which completes the proof that $\frac{d}{dx} \sin x = \cos x$.

2. DERIVATIVE OF y = cos x

2.1 Introduction

Now, we know that $\frac{d}{dx} \sin x = \cos x$ where x is any real number and we take the sine of the angle whose radian measure is x. By the Chain Rule

$$\frac{d}{dx} \sin u = \frac{d}{du} \sin u \cdot \frac{du}{dx}$$

where u is a differentiable function of x. We use the Chain Rule to obtain derivatives for the other trigonometric functions.

2.2 Proof

We will now prove that we were correct in our guess about the derivative of y = cos x. We use the identities

$$\cos x = \sin \left(\frac{\pi}{2} - x\right)$$

and

$$\cos (x - y) = \cos x \cos y + \sin x \sin y.$$

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sin \left(\frac{\pi}{2} - x\right)$$

$$= \cos \left(\frac{\pi}{2} - x\right) \cdot \frac{d}{dx} \left(\frac{\pi}{2} - x\right) = \cos \left(\frac{\pi}{2} - x\right) \cdot (-1)$$

$$= \left(\cos \frac{\pi}{2} \cdot \cos x + \sin \frac{\pi}{2} \cdot \sin y\right) (-1)$$

$$= (0 \cdot \cos x + 1 \cdot \sin x) (-1) = -\sin x.$$

3. WHEN DEGREE MEASURE IS USED

3.1 Geometric Consideration

Let us now attempt to resolve the problem that arose in Unit 159 where we computed the derivative of y = sin x at x = 35°. We hoped to get cos 35°. Let us use the notation sin x° if we are taking the sine of the angle whose degree is x and the notation sin x or sin x (radians) if we are taking the sine of the angle whose radian measure is x. We consider the following graphs. Observe in Figure 2 that -0.801 the value of cos 2.5 does not seem to disagree with what the slope of the tangent to y = sin x at x = 2.5 looks to be. Now, in Figure 3, does the slope of the tangent to y = sin x at x = 2.5 appear to be 0.9996? It should be if $\frac{d}{dx} \sin x° = \cos x°$. Now we see geometrically that we should not expect the result we obtained when we were using values of trigonometric functions whose radian measure was x.

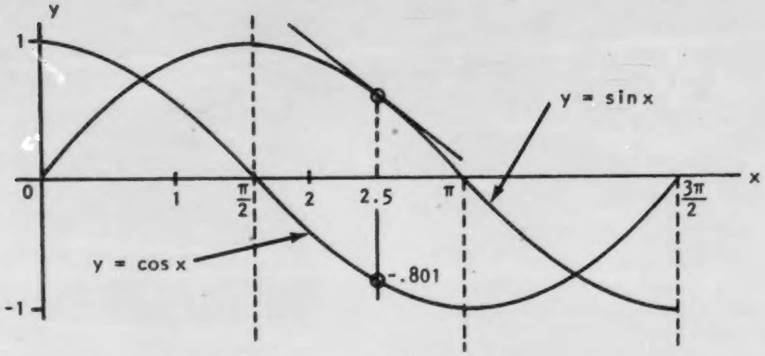

Figure 2. Radian measure.

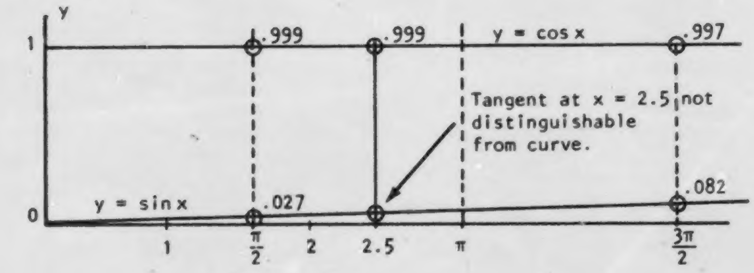

Figure 3. Degree measure.

3.2 Obtaining a Formula When the Angle is Measured in Degrees

Let us obtain a formula for

$$\frac{d(\sin x°)}{dx} .$$

Where u is a differentiable function of x, the Chain Rule gives

$$\frac{d[\sin u \text{ (radians)}]}{dx} = \cos u \text{ (radians)} \cdot \frac{du}{dx} .$$

Now, $\sin x° = \sin (\frac{\pi}{180} x)$.

Thus,

$$\frac{d(\sin x°)}{dx} = \frac{d}{dx}\left[\sin (\frac{\pi}{180} x)\right]$$

$$= \cos (\frac{\pi}{180} x) \cdot \frac{\pi}{180} = \cos x° \cdot \frac{\pi}{180}$$

by Chain Rule.

3.3 Reconsiderations

Returning to our geometric consideration of $y = \sin x°$, where $x = 2.5$ we have

$$\frac{d(\sin 2.5°)}{dx} = \cos 2.5 \cdot \frac{\pi}{180}$$

$$= (0.9990)(\frac{3.1416}{180}) = 0.0174 .$$

This certainly looks like a much more believable value for the slope of the tangent line at x = 2.5 which is sketched in Figure 3.

We now can express $\frac{d(\sin 35°)}{dx}$ in terms of cos 35°.

$$\frac{d(\sin 35°)}{dx} = \cos 35° \cdot \frac{\pi}{180}$$

$$= 0.8192\ (\frac{3.1416}{180}) = 0.0143.$$

Our approximation of 0.0143 that we obtained in Unit 159 on page 12 now looks good.

From now on when we differentiate trigonometric functions we will always use raidan measure.

4. PRACTICE PROBLEMS INVOLVING sin u AND cos u

4.1 Finding Derivatives

Let us combine our new knowledge with previous techniques for finding derivatives to work some problems. By the Chain Rule, where u is a differentiable function of x, we have

$$\frac{d}{dx}\sin u = \frac{d}{du}\sin u \cdot \frac{du}{dx} = \cos u \cdot \frac{du}{dx}$$

and

$$\frac{d}{dx}\cos u = \frac{d}{du}\cos u \cdot \frac{du}{dx} = -\sin u\,\frac{du}{dx}\,.$$

These formulas are used in the following examples.

Example 1: $\frac{d}{dx}\sin 3x = 3\cos 3x$

Example 2: $\frac{d}{dx}\sin(x^2 + 1) = 2x\cos(x^2 + 1)$

Example 3: $\frac{d}{dx}\cos(2x - 3) = -2\sin(2x - 3)$

Exercises

For each of the following find $\frac{dy}{dx}$.

1. $y = \sin 2x^2$
2. $y = \cos 2x$
3. $y = \cos(x^2 - x)$
4. $y = \sin(x/3)$
5. $y = \cos x^\circ$

In the following examples, we use the formulas for taking derivatives where a sum, product or quotient is also involved. These formulas are given in Appendix 3 if you need to review them.

Example 1: $y = \sin 2x + \cos x$

$$\frac{dy}{dx} = 2\cos 2x - \sin x.$$

Example 2: $y = 2\sin\cos x$

$$\frac{dy}{dx} = 2\sin x(-\sin x) + \cos x(\cos x)$$

$$\frac{dy}{dx} = 2[(-\sin^2 x + \cos^2 x)].$$

Example 3: $y = \frac{\sin 2x}{1 + \cos x}$

$$\frac{dy}{dx} = \frac{(1 + \cos x)\ 2\cos 2x - (\sin 2x)(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{2(1 + \cos x)\cos 2x + \sin 2x\sin x}{(1 + \cos x)^2}.$$

Exercises For each of the following find $\frac{dy}{dx}$.

6. $y = \sin x + \cos x$
7. $y = x^2\cos 2x$
8. $y = \cos^3(2x)$
9. $y = \sin x + x$
10. $y = \cos 2x - 2\cos x$
11. $y = \sin^2 x\cos^2 x$

4.2 Finding Antiderivatives

We recall from our work with antiderivatives that

$$\int f(u) = F(u) + c$$

where

$$\frac{d}{du}F(u) = f(u).$$

Thus,

$$\int \cos u\,du = \sin u + c$$

since

$$\frac{d}{du}\sin u = \cos u$$

and

$$\int -\sin u\,du = \cos u + c$$

since

$$\frac{d}{du}\cos u = -\sin u.$$

The following are examples using these formulas plus the formulas

$$a\int f(u)\,du = \int a\,f(u)\,du$$

and

$$\int f(u) + g(u)\,du = \int f(u)\,du + \int g(u)\,du.$$

Example 1: $\int \sin x\,dx = -\int -\sin x\,dx = -\cos x + c.$

Example 2: $\int \cos 2x\,dx = \frac{1}{2}\int \cos 2x\,(2dx)$

$= \frac{1}{2}\sin 2x + c.$

Example 3: $\int (2 - \sin\frac{1}{2}x)\,dx = \int 2\,dx + \int -\sin\frac{1}{2}x\,dx$

$= \int 2\,dx + 2\int -\sin(\frac{1}{2}x)\cdot(\frac{1}{2}dx)$

$= 2x + 2\cos\frac{1}{2}x + c.$

Exercises

12. $\int \cos(-2x)\,dx$
13. $\int \sin(\frac{x}{2})\,dx$
14. $\int 2\cos x\,dx$
15. $\int 2\sin 2x\,dx$
16. $\int 3\cos(\frac{x}{3})\,dx$
17. $\int \frac{1}{4}\sin(\frac{x}{4})\,dx$
18. $\int (3 - \sin x)\,dx$
19. $\int (\cos x + \sin 2x)\,dx$
20. $\int \sin^2 x\,dx$
21. $\int \cos^2 2x\,dx$

Hint for 20 and 21: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x).$

---

5. CHALLENGE PROBLEMS REVISITED

5.1 Introduction to Solutions

Now we are armed with new knowledge and skills in calculus where sin u and cos u are involved. With this

additional ammunition let us return to battle with the challenge problems hoping for a successful outcome.

If you have trouble getting started on a problem or hit a snag refer to the discussion for help and then try to continue on your own. After completing your solution to a challenge problem compare your work with the solution given.

5.2 Out Fishing Again

Our first step is to use the information from the graph in Figure 1 of Unit 158 to determine the equation for the position of Jack's cork.

From our knowledge of trigonometry it is clear that we have the graph of an equation of the form $y = A \sin Bx$, where A is the amplitude and $\frac{2\pi}{B}$ is the period. Thus, from observing Figure 1 of Unit 158, we see that $A = 4$ and $\frac{2\pi}{B} = 1.6$ or $B = \frac{5\pi}{4}$. Substituting, $y = 4 \sin \frac{5\pi}{4} t$ is the equation for the position of the cork. The position of the cork at $t = 0.5$ sec. is

$$y = 4 \sin \left[\frac{5\pi}{4}(1.5)\right] = 3.969 \text{ cm.}$$

At $t = 1.5$ sec. we have

$$y = 4 \sin \left[\frac{5\pi}{4}(1.5)\right] = -1.531 \text{ cm.}$$

Now, the speed (or instantaneous rate of change of position with respect to change in time) is $\frac{dy}{dt}$. Thus, we have

$$\frac{dy}{dt} = 4\left(\frac{5\pi}{4}\right)\cos\frac{5\pi}{4}t = 5\pi\cos\frac{5\pi}{4}t.$$

So, the speed at $t = 0.5$ sec. is

$$\frac{dy}{dt}_{(t=.5)} = 5\pi\cos\left[\frac{5\pi}{4}(.5)\right] = -6.011 \text{ cm/sec.}$$

and the speed at $t = 1.5$ sec is

$$\frac{dy}{dt}_{(t=1.5)} = 5\pi \cos\left[\frac{5\pi}{4}(1.5)\right] = 14.512 \text{ cm/sec.}$$

The last question to be answered is "At what point during the 2 second interval is the cork falling at the fastest rate and what is the rate?" We wish to minimize $\frac{dy}{dt} = 5\pi \cos \frac{5\pi}{4} t$ on the interval (0,2). The minimum value of $\cos \frac{5\pi}{4} t$ on (0,2) is -1 which is attained at $\frac{5\pi}{4} t = \pi$ or $t = 0.8$ sec. This will minimize $\frac{dy}{dt}$. Thus

$$\frac{dy}{dt}_{(\min)} = 5\pi(-1) = -15.708 \text{ cm/sec.}$$

5.3 Maximum Putter Gutter

Referring to Figure 2 of Unit 158, we see that if we maximize the cross-section we will maximize the capacity of the gutter. We also observe that θ can be chosen so that $\theta \leq \frac{\pi}{2}$.

From Figure 2 of Unit 158 we have $\csc\theta = \frac{x}{4}$ or $x = 4 \csc\theta$ and $\cot\theta = \frac{y}{4}$ or $y = 4\cot\theta$.
A(Area of cross section)

$$\begin{aligned}
&= \text{Area of Rectangle} - \text{Area of Triangle}\\
&= 4[(8 - x) + y] - \tfrac{1}{2}(4)\, y\\
&= 4[(8 - 4\csc\theta) + 4\cot\theta] - \tfrac{1}{2}(4\cot\theta)\cdot 4 \quad \text{(substituting for x and y)}\\
&= 8(4 - 2\csc\theta + 2\cot\theta - \cot\theta)\\
&= 8(4 - 2\csc\theta + \cot\theta).
\end{aligned}$$

The area will be maximum when $\frac{da}{d\theta} = 0$. We don't know how to find the derivatives of $\csc\theta$ and $\cot\theta$ at this time. What we can do is express $\csc\theta$ and $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$. Doing this we have

$$A = 8\left(4 - \frac{2}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$\begin{aligned}
\frac{dA}{d\theta} &= 8\left[-\left(\frac{\sin\theta\,(0) - 2\cos\theta}{\sin^2\theta}\right) + \left(\frac{\sin\theta\,(-\sin\theta) - \cos\theta(\cos\theta)}{\sin^2\theta}\right)\right]\\
&= 8\left[\frac{2\cos\theta}{\sin^2\theta} + \frac{(-1)(\sin^2\theta + \cos^2\theta)}{\sin^2\theta}\right]\\
&= 8\left[\frac{2\cos\theta - 1}{\sin^2\theta}\right],
\end{aligned}$$

since $\sin^2\theta + \cos^2\theta = 1$. For $\frac{dA}{d\theta} = 0$ we must have the expression within brackets equal to zero, but this means that the numerator must be zero. Thus, $2\cos\theta - 1 = 0$ or $\cos\theta = \frac{1}{2}$. Since $\theta \leq \frac{\pi}{2}$ we have $\theta = \frac{\pi}{3}$ or 60°. We also observe that for θ such that $0 \leq \theta \leq \frac{\pi}{3}$ we have $\cos\theta > \frac{1}{2}$ which implies $2\cos\theta - 1 > 0$. Now looking at our expression we note that when $2\cos\theta - 1 > 0$ we have $\frac{dA}{d\theta} > 0$. In a similar way we can conclude that $\frac{dA}{d\theta} < 0$ when $\frac{\pi}{3} < \theta < \frac{\pi}{2}$.

We have now verified that $\theta = \frac{\pi}{3}$ is in fact the value of θ for which the θ area is maximum.

Since $x = 4\csc\theta$, we have

$$x = 4\csc\frac{\pi}{3} = 4.619$$

and

$$8 - x = 3.381.$$

Thus, the bend should be at approximately 3.381 inches from the bend for the right angle and the metal should be bent up $\frac{\pi}{3}$ radians or 60° for a gutter with maximum capacity.

5.4 Average Power Computed

We have (Average power)

$$P = \frac{1}{T}\int_0^T p\, dt,$$

where

$$p = 1700 \sin^2 \frac{\pi}{12} t.$$

Observing Figure 5 of Unit 158, we see that the graph of p repeats itself every 12 micro-seconds. Thus T = 12. Substituting, gives

$$P = \frac{1}{12}\int_0^{12} 1700 \sin^2 \frac{\pi}{12} t\,dt = \frac{1700}{12}\int_0^{12} \sin^2 \frac{\pi}{12} t\,dt.$$

This problem is like Exercise 20 in section 4.3. We will replace

$$\sin^2 \frac{\pi}{12} t$$

by

$$\tfrac{1}{2}(1 - \cos \frac{\pi}{6} t)$$

Thus

$$P = \frac{1700}{12}\int_0^{12} \tfrac{1}{2}(1 - \cos \frac{\pi}{6} t)\,dt$$

$$= \frac{1700}{24}\left[\int_0^{12} dt - \int_0^{12} \cos \frac{\pi}{6} t\,dt\right]$$

$$= \frac{1700}{24}\left[\int_0^{12} dt - \frac{6}{\pi}\int_0^{12} \frac{\pi}{6} \cos \frac{\pi}{6} t\,dt\right]$$

$$= \frac{1700}{24}\left\{t\Big]_0^{12} - \frac{6}{\pi}\left(\sin \frac{\pi}{6} t\Big]_0^{12}\right)\right\}$$

$$= \frac{1700}{24}\left[(12 - 0) - \frac{6}{\pi}(\sin 2\pi - \sin 0)\right]$$

$$= \frac{1700}{24}(12) = \frac{1700}{2} = 850 \text{ watts.}$$

Now, consider the geometric interpretation,

$$\int_0^{12} \sin^2 \frac{\pi}{12} t\,dt$$

represents the area (A) under one hump of the curve shown by //// in Figure 4. We know $\frac{1}{12}$ A = 850, so

A = 12(850). Thus the rectangle shown by \\\\ with height equal to the average power P has the same area as the area under the hump.

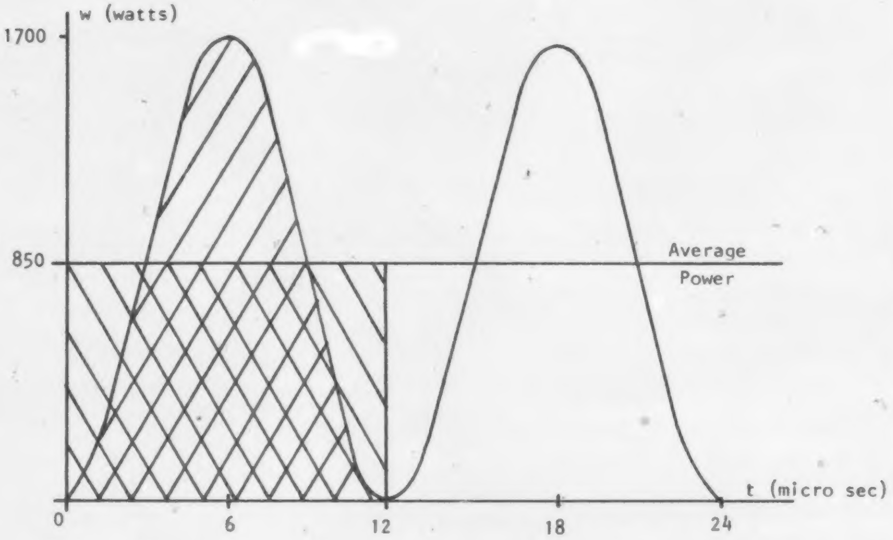

Figure 4. A geometric interpretation of average power.

5.5 Pulling a Box Correctly

Finding $\frac{dF}{d\theta}$ using the quotient rule gives

$$\frac{dF}{d\theta} = \frac{-KW\,(K \cos\theta - \sin\theta)}{(K \sin\theta + \cos\theta)^2}$$

if $\frac{dF}{d\theta} = 0$, we must have

$$K \cos\theta - \sin\theta = 0$$

or

$$K = \frac{\sin\theta}{\cos\theta}.$$

Thus, the first derivative is zero for θ such that $\tan\theta = k$ we can also show the first derivative changes from negative to positive at this value of θ which verifies that this is the angle where the force is minimum. Now, K is always some positive number, so Jason is correct and the angle depends on the coefficient of friction.

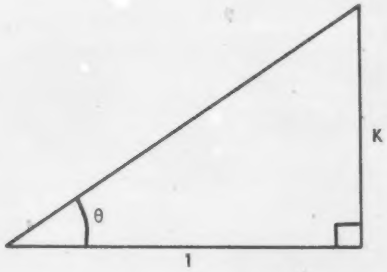

Figure 5. Coefficient of friction.

If one object has a higher coefficient of friction than another object on a particular surface then the angle at which the force should be applied is greater.

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6. MODEL EXAM

1. What technique is used to show $\lim_{h\to 0} \frac{\sin h}{h} = 1$?

2. What limit other than $\lim_{h\to 0} \frac{\sin h}{h} = 1$ was used to prove $\frac{d}{dx}(\sin x) = \cos x$?

Complete the statements in 3 and 4.

3. The value of $\frac{d}{dx}\sin x$ is ________________,when $x = 17°$.

4. The value of $\frac{d}{dx}\cos x$ is ________________,when $x = 39°$

In problems 5 thru 10, find $\frac{dy}{dx}$.

5. $y = \sin(x^2 - 3)$
6. $y = 2\sin x + \cos 3x$
7. $y = \sin^2 x$
8. $y = \sin x \cos x$
9. $y = \cos(3x^2 - x)$
10. $y = \cos^2 x \sin x$

Find the antiderivatives in Problems 11 thru 13.

11. $\int \sin x \, dx$
12. $\int \cos(3x)\, dx$
13. $\int (5 - 2\sin x)\, dx$

14. Given $F = \frac{KW}{K\sin\theta + \cos\theta}$, find $\frac{dF}{d\theta}$.

7. ANSWERS TO EXERCISES

1. $y = \sin 2x^2$

   $\frac{dy}{dx} = 4x \cos 2x^2$

2. $y = \cos 2x$

   $\frac{dy}{dx} = -2 \sin 2x$

3. $y = \cos (x^2 - x)$

   $\frac{dy}{dx} = -(2x - 1) \sin (x^2 - x)$

4. $y = \sin \frac{x}{3}$

   $\frac{dy}{dx} = \frac{1}{3} \cos \frac{x}{3}$

5. $y = \cos x° = \cos (\frac{\pi}{180} x)$

   $\frac{dy}{dx} = -\sin (\frac{\pi}{180} x) \cdot \frac{\pi}{180}$

6. $y = \sin x + \cos x$

   $\frac{dy}{dx} = \cos x - \sin x$

7. $y = x^2 \cos 2x$

   $\frac{dy}{dx} = -2x^2 \sin 2x + 2x \cos 2x$

   $\frac{dy}{dx} = -2x (x \sin 2x - \cos 2x)$

8. $y = \cos^3 (2x)$

   $\frac{dy}{dx} = -6 \cos^2 (2x) \sin (2x)$

9. $y = \sin x + x$

   $\frac{dy}{dx} = \cos x + 1$

10. $y = \cos 2x - 2 \cos x$

    $\frac{dy}{dx} = -2 \sin 2x + 2 \sin x$

    $\frac{dy}{dx} = 2(\sin x - \sin 2x)$

11. $y = \sin^2 x \cos^2 x$

    $= -2 \sin^2 x \cos x \sin x + 2 \cos^2 x \sin s \cos x$

    $= 2 (\cos^3 x \sin x - \sin^3 x \cos x)$

    $= 2 \cos x \sin x (\cos^2 x - \sin^2 x)$

12. $-\frac{1}{2}\int -2 \cos (-2x)\, dx = -\frac{1}{2} \sin (-2x) + c$

13. $-2\int -\frac{1}{2} \sin (\frac{x}{2})\, dx = -2 \cos (\frac{x}{2}) + c$

14. $2\int \cos x\, dx = 2 \sin x + c$

15. $-\int -2 \sin 2x\, dx = -\cos 2x + c$

16. $9\int \frac{1}{9} \cdot 3 \cos (\frac{x}{3})\, dx = 9 \sin (\frac{x}{3}) + c$

17. $-\int -\frac{1}{4} \sin (\frac{x}{4})\, dx = -\cos (\frac{x}{4}) + c$

18. $\int (3 - \sin x)\, dx = \int 3\, dx + \int -\sin\, dx = 3x + \cos x + c$

19. $\int (\cos x + \sin 2x)\, dx = \int \cos x\, dx - \frac{1}{2}\int -2 \sin 2x\, dx$

    $= \sin x - \frac{1}{2} \cos 2x + c$

20. $\int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$

    $= \frac{1}{2}[\int dx - \frac{1}{2}\int 2 \cos 2x\, dx] = \frac{x}{2} - \frac{1}{4} \sin 2x + c$

21. $\int \cos^2 2x\, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$

    $= \frac{1}{2}[\int dx + \frac{1}{4} \int 4 \cos 4x\, dx]$

    $= \frac{x}{2} + \frac{1}{8} \sin 4x + c$

UNIT 161: DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS

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5. ANSWERS TO MODEL EXAM 10

1. OBTAINING FORMULAS

1.1 Introduction

In our discussion of the solution to the Putter Gutter Problem (see Section 5.3 in Unit 160) we obtained the cross sectional area A of the gutter from the equation $A = 8(4 - 2\csc\theta + \cot\theta)$. To obtain $\frac{dA}{d\theta}$ we had to express $\csc\theta$ and $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$ since we did not have formulas for the derivatives of these trigonometric functions. To eliminate this extra work in the future, we will now derive the formulas for the derivatives of the other four trigonometric functions. We will follow the same plan used to find $\frac{dA}{d\theta}$ and express each of the other functions in terms of the sine and cosine functions.

1.2 The Derivative of tan x

To obtain the formula for the derivative of $\tan x$ we express $\tan x$ as $\frac{\sin x}{\cos x}$ and use the rule for quotients which guarantees that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,(du/dx) - u\,(du/dx)}{v^2}$$

where u and v are differentiable functions of x.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

1.3 The Derivative of sec x

We will express $\sec x$ as $\frac{1}{\cos x}$ and use the quotient rule again to obtain the derivative of $\sec x$.

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x \cdot 0 - 1\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x.$$

Exercises

1. Expressing cot x as $\frac{\cos x}{\sin x}$, follow the procedure used to obtain the derivative of tan x to obtain the derivative of cot x.
2. Find the derivative of csc x by replacing csc x by $\frac{1}{\sin x}$ and using the same technique as used for the derivative of sec x.

1.4 Complete List

With the solutions to Exercise 1 and Exercise 2 in the preceding section we now have obtained formulas for the derivatives of all six of the trigonometric functions of x. You should check your procedures and results to Exercises 1 and 2 with the solutions given at the end of this unit. The formulas for the derivatives of the six trigonometric functions of x are listed in Appendix 4 for your convenience.

2. PRACTICE FINDING DERIVATIVES

2.1 Using Chain Rule Again

Having the formulas for the derivative of tan x, cot x, sec x and csc x, we wish to obtain the formulas for these trigonometric functions of u where u is a differentiable function of x. We apply the Chain Rule in each case just as we did for sin u and cos u. For tan u we have

$$\frac{d}{dx}\tan u = \frac{d}{du}\tan u \cdot \frac{du}{dx} = \sec^2 u\,\frac{du}{dx}\,.$$

The formulas for the derivatives of the remaining trigonometric functions of u follow in the same manner. The formulas for all six trigonometric functions of u are listed in Appendix 4 for your reference.

2.2 Applying the Formulas

These formulas are applied in the following examples.

Example 1: $\frac{d}{dx}\tan\left(\frac{x}{3}\right) = \frac{1}{3}\sec^2\frac{x}{3}.$

Example 2: $\frac{d}{dx}\csc 2x = -2\csc 2x \cot 2x.$

Example 3: $\frac{d}{dx}\cot x^2 = -2x\csc x^2.$

Exercises

Find $\frac{dy}{dx}$ in each of the following.

3. $y = \cot\frac{x}{2}.$
4. $y = \csc(2x - 1).$
5. $y = \tan x^3.$
6. $y = \sec(-x).$

2.3 More Involved Applications

In the following examples a sum, product or quotient may also be involved. These formulas are listed in Appendix 3 for reference. Derivatives involving sin u and cos u which we considered previously are included in the examples and exercises that follow.

Example 1: $y = \tan 2x + \sec x$

$$\frac{dy}{dx} = 2\sec^2 2x + \sec x \tan x.$$

Example 2: $y = \tan 2x.$

$$\frac{dy}{dx} = x\ (2 \sec^2 2x) + \tan 2x$$

$$= 2x \sec^2 2x + \tan 2x.$$

Example 3: $y = \dfrac{\cos x}{1 + \tan 2x}.$

$$\frac{dy}{dx} = \frac{(1 + \tan 2x)(-\sin x) - 2\ \text{cox}\ x(\sec^2 2x)}{(1 + \tan 2x)^2}.$$

Exercises

For each of the following find $\frac{dy}{dx}$.

7. $y = \sin x + \tan x.$
8. $y = x^2 \csc 2x.$
9. $y = \tan x \sin 2x.$
10. $y = \cot^3 (2x).$
11. $y = \sec x + x.$
12. $y = \cos 2x - 2 \csc x.$
13. $y = x^2 \tan (\frac{x}{2}).$
14. $y = \dfrac{\sin 2x}{1 - \tan x}.$

2.4 Puttering Around

We return to the Putter Gutter Problem of Unit 158 one more time. Let us find $\frac{dA}{d\theta}$ now that we have the formulas for the derivatives of csc θ and cot θ.

$$A = 8\ (4 - 2 \csc \theta + \cot \theta).$$

$$\frac{dA}{d\theta} = 8 \left[-2(-\csc \theta \cot \theta) + (-\csc^2 \theta)\right]$$

$$= 8\ [2 \csc \theta \cot \theta - \csc^2 \theta],$$

We see that having the formulas for the derivatives of all the trigonometric functions available made finding $\frac{dA}{d\theta}$ easier, and they may be of use to us in future putterings.

3. MODEL EXAM

Differentiate the functions in Problems 1 and 2 by first expressing them in terms of sin x and cos x.

1. $\frac{d}{dx}(\tan x) =$

2. $\frac{d}{dx}(\csc x) =$

Find $\frac{dy}{dx}$ in each of the following problems.

3. $y = \cot x$.
4. $y = \sec x$.
5. $y = \sec(3x + 5)$.
6. $y = \sin 2x + \tan x^2$.
7. $y = x \tan x$.
8. $y = \frac{\cos x}{1 + \tan 2x}$.

4. ANSWERS TO EXERCISES

1. $\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$. By quotient rule,

$$\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x\,(-\sin x) - \cos x\,(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1\,(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1\,(1)}{\sin^2 x} = \csc^2 x.$$

2. $\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$. By quotient rule,

$$\frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x\,(0) - 1\,(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

$$= \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x.$$

Note: We essentially did these exercises in our calculations for $\frac{dA}{d\theta}$ in 5.3 of Unit 160.

3. $y = \cot \frac{x}{2}$

$$\frac{dy}{dx} = \left(-\csc^2 \frac{x}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}\csc^2 \frac{x}{2}.$$

4. $y = \csc(2x - 1)$.

$$\frac{dy}{dx} = -\csc(2x - 1)\cot(2x - 1)\,2$$

$$= -2\csc(2x - 1)\cot(2x - 1).$$

5. $y = \tan x^3$.

$$\frac{dy}{dx} = (\sec^2 x^3)(3x^2) = 3x^2 \sec^2 x^3.$$

6. $y = \sec(-x).$

$$\frac{dy}{dx} = [\sec(-x)\tan(-x)](-1) = -\sec(-x)\tan(-x).$$

7. $y = \sin x + \tan x.$

$$\frac{dy}{dx} = \cos x + \sec^2 x.$$

8. $y = x^2 \csc 2x.$

$$\frac{dy}{dx} = x^2 [-\csc(2x)\cot(2x)(2)] + [\csc(2x)]2x$$

$$= 2x \csc(2x)(-x \cot 2x + 1).$$

9. $y = \tan x \sin 2x.$

$$\frac{dy}{dx} = \tan x \cos(2x)(2) + \sin(2x)\sec^2 x.$$

10. $y = \cot^3(2x).$

$$\frac{dy}{dx} = 3\cot^2(2x)[-\csc^2(2x)](2)$$

$$= -6\cot^2(2x)\csc^2(2x).$$

11. $y = \sec x + x.$

$$\frac{dy}{dx} = \sec x \tan x + 1.$$

12. $y = \cos 2x - 2\csc x.$

$$\frac{dy}{dx} = -\sin(2x)(2) - 2(-\csc x \cot x)$$

$$= -2\sin(2x) + 2\csc x \cot x.$$

13. $y = x^2 \tan(\frac{x}{2}).$

$$\frac{dy}{dx} = x^2 \sec^2(\tfrac{x}{2})(\tfrac{1}{2}) + \tan(\tfrac{x}{2}) \cdot 2x$$

$$= x\left[\frac{x}{2}\sec^2(\tfrac{x}{2}) + 2\tan(\tfrac{x}{2})\right].$$

14. $y = \dfrac{\sin 2x}{1 - \tan x}.$

$$\frac{dy}{dx} = \frac{(1 - \tan x)\cos(2x)(2) - \sin(2x)(-\sec^2 x)}{(1 - \tan x)^2}$$

$$= \frac{2(1 - \tan x)\cos(2x) + \sin(2x)\sec^2 x}{(1 - \tan x)^2}.$$

APPENDIX 1
THE TANGENT METHOD FOR ESTIMATING DERIVATIVES*

Objective 1: *To be able to use a triangle and ruler to estimate the slope of a line on a graph.*

Graphical Method for Finding the Slope of a Line

Many times in our work we want to measure the slopes of lines plotted on graphs. We can always calculate the slope of a line by reading the coordinates of two points on the line and applying the formula

$$\text{slope} = \frac{\text{change in vertical units}}{\text{change in horizontal units}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

But there is an easier way that saves the effort of reading the four numbers from the graph necessary to calculate each slope. For this method, you will need a straight edge or ruler, and a small drawing triangle.

Figure 1.1 is a graph that shows the profile of the Union Pacific Railroad. The problem is to find the slope of the railroad between *Lakeside, Utah* and *Wells, Nevada* directly from the graph using as little arithmetic as possible. The following steps provide an easy method to measure this slope.

Step 1 (see Figure 1.2). *Place the triangle* with one edge along the line whose slope you wish to measure.

Step 2 (see Figure 1.3). *Place the ruler* against the other side of the triangle. Check that the first edge of the triangle is still along the line you wish to measure.

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*Adapted by the UMAP Project Staff from Differentiation, Second Edition, 1975, Project CALC, Education Development Center, Inc., Newton, Massachusetts, pp. 27-60.

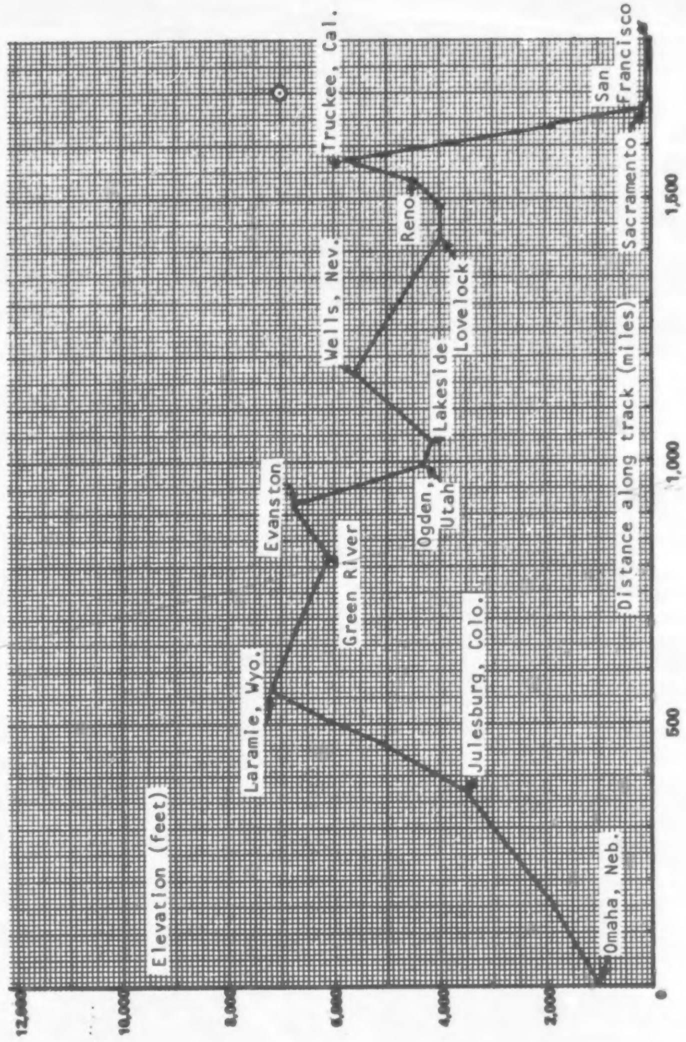


Figure 1.1. Profile of the Union Pacific - Southern Pacific R.R.

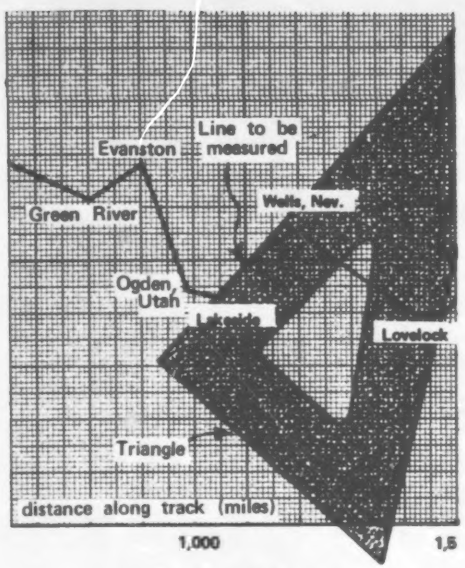

Line to be measured
Evanston
Green River
Wells, Nev.
Ogden, Utah
Lakeside
Lovelock
Triangle
distance along track (miles)
1,000
1,5

Figure 1.2.

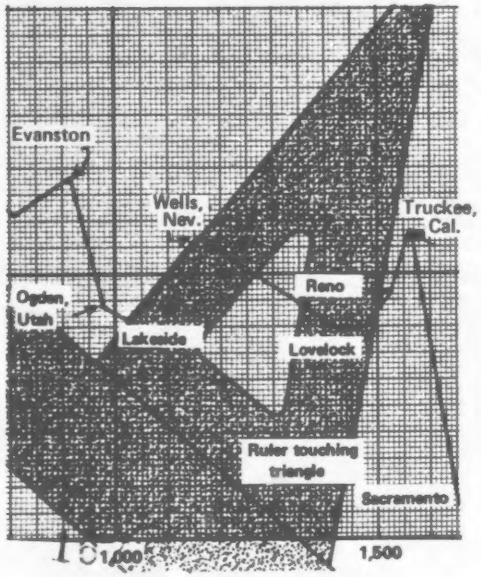

Evanston
Wells, Nev.
Truckee, Cal.
Reno
Ogden, Utah
Lakeside
Lovelock
Ruler touching triangle
Sacramento
1,000
1,500

Figure 1.3.

Step 3 (see Figure 1.4). *Slide the triangle* along the ruler (holding the ruler firmly so it will not slip) until the first edge of the triangle passes through an easily read intersection of the graph paper. (In this example, the triangle was moved until its edge passed through the intersection of the 1,000-mile line with the 5,000-foot line. Since the triangle was slid along the ruler, this edge is still parallel to the line whose slope is to be measured. The slope of the edge of the triangle is therefore still the same as the slope of the original line.)

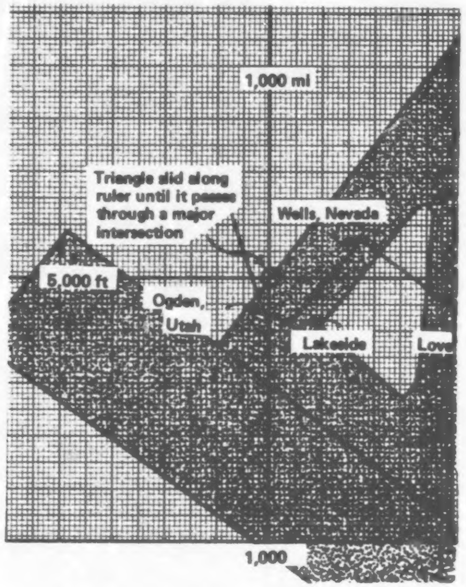

1,000 mi
Triangle slid along ruler until it passes through a major intersection
Wells, Nevada
5,000 ft
Ogden, Utah
Lakeside
Love
1,000

Figure 1.4.

Step 4 (see Figure 1.5). *Read the slope* of the edge of the triangle at the point where the triangle cuts the next major vertical line on the graph paper. Here the next major vertical line is at 1,100 miles or is 100 miles further from the first easily read intersection. The triangle edge intersects this line 1,100 feet above the first read intersection. The upward slope of the triangle, and of the track is therefore

$$\frac{1{,}100 \text{ feet}}{100 \text{ miles}} = 11 \frac{\text{feet}}{\text{mile}}$$

or 11 feet per mile. Note that if you choose the horizontal distance to be 1, 10, 100 or 1,000 miles, the division can be easily done in your head.

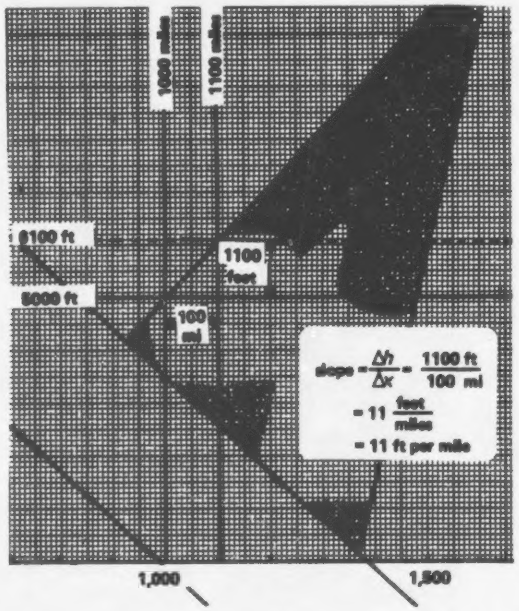


Figure 1.5.

Exercise 1 (Short Method for Grades on the UP-SP RR). Using the method above, find the slope of the track between

Reno, NV and Truckee, CA ______
Green River and Evanston, WY ______
Midsection between Omaha, NE
and Julesburg, CO ______
Wells and Lovelock, NV ______

Improved Method for Finding Slopes

It is possible to improve this method to avoid the division and the placing of the decimal point. Try the improved method on the same graph of the UP-SP RR you have been using (Figure 1.1). As with many "how to do it" directions, it takes much longer to describe than to do, so follow along and your patience will be rewarded. If you have trouble following the directions, have your instructor give you a quick demonstration.

Setting Up a Scale for Reading Slopes

Step A (Figure 1.6). *Mark a standard intersection* one major division in from the right-hand edge of the graph paper. (Such an intersection has already been marked with a ⊙ in Figure 1.1 at the beginning of this section, and in which you may continue to set up a slope scale and make measurements.)

Step B (Figure 1.6). *Temporarily mark* the first major division above the center of the slope scale with the number of vertical units it separates. Here it is marked +1,000 feet since it represents an increase in elevation of this amount.

Step C. *Calculate the value of the slope* for this first division by taking the ratio of the vertical increase (1,000 feet) to the horizontal increase (100 miles) for one major division.

$$\text{slope} = \frac{\Delta h}{\Delta x} = \frac{1{,}000 \text{ feet}}{100 \text{ miles}} = 10 \frac{\text{ft}}{\text{mi}}$$

or 10 feet per mile. Write this number of the scale in place of the temporary mark of 1,000 feet.

Step D. *Mark the slope values* +10, +20, +30, etc., opposite the main divisions going upward from zero. Place -10, -20, -30, etc., opposite the main divisions going downward from zero. (See Figure 1.7.) Write the units in which the slope is measured (ft/mi) at the top of the scale. You are now ready to use the scale to measure slope.

To set up a scale for direct reading of slopes . . .

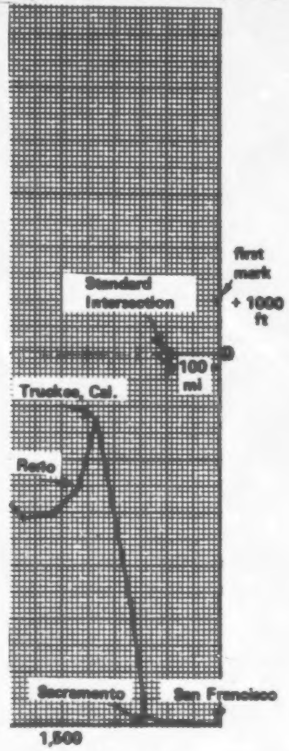


Figure 1.6.

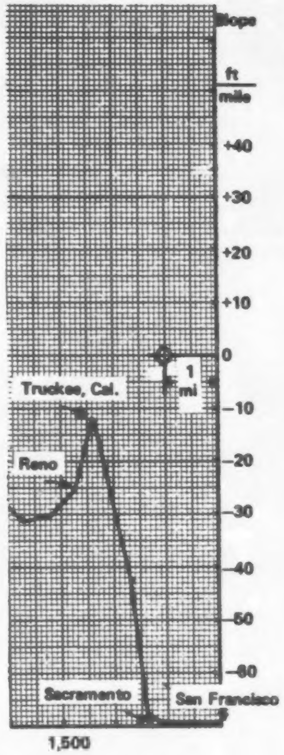


Figure 1.7.

Measuring the Slope with the Slope Scale

To use the scale just marked to measure slopes in a convenient and direct way, set the triangle to the line you wish to measure and slide it by means of the ruler until its edge passes through the standard intersection ⊙ . Read the value of the slope at the point where the edge of the triangle crosses the slope scale line. (See Figure 1.8.)

NOTE: You cannot always slide the triangle to a position where its edge passes through both the standard intersection and the slope scale in one slide along the ruler. When this happens, hold the triangle firmly in the last position, shift the ruler so it is along a different

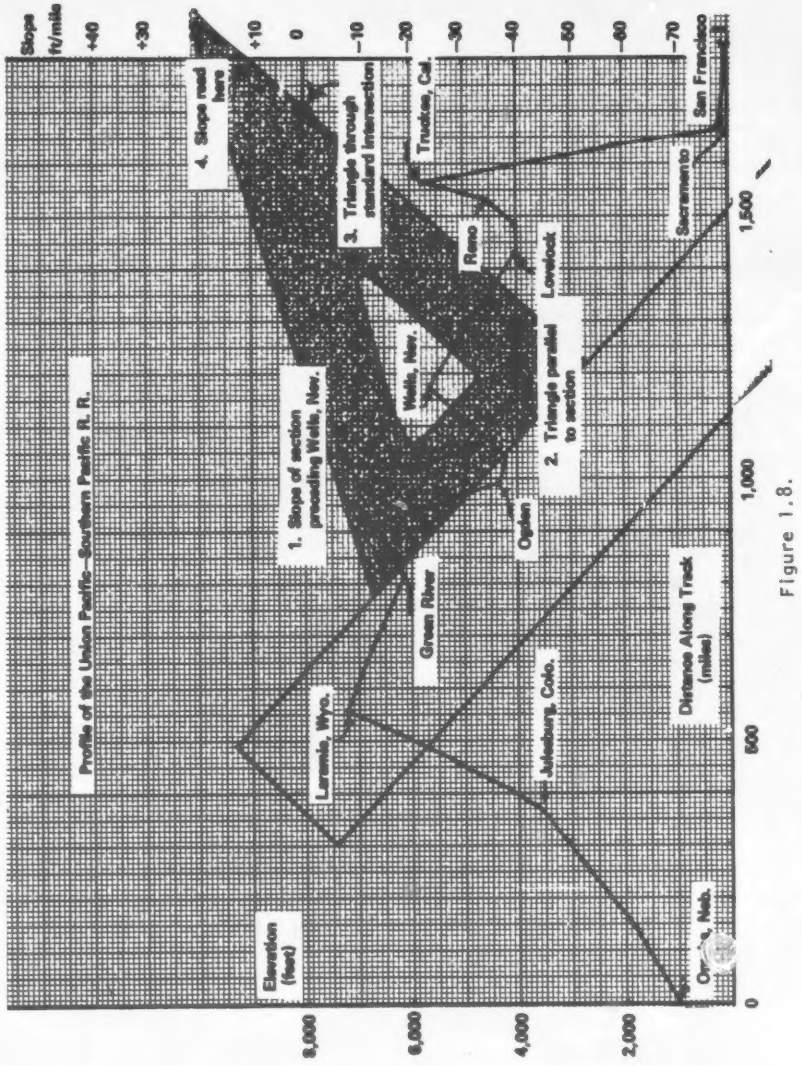


Figure 1.8.

edge of the triangle, and then proceed to slide the triangle in a new direction. By a series of such parallel slides of the triangle, it is possible to position the triangle so that the slope may be read.

Exercise 2 (Direct Method for Grades on the UP-SP Railroad). For practice, recheck the slopes you measured before and then try these:

Section	Slope
1st section after Julesberg	________
1st section before Green River	________
1st section after Truckee	________

Objective 2: *To be able to calculate numerical values of the average rate of change between values of a function: (a) from a table of values, and (b) from a graph.*

A function in mathematics is a rule or a recipe that relates two quantities such as distance and time. We now begin a process that will lead eventually to a method for calculating the rate of change of a smoothly varying function. As an example we use the curve shown in Figure 1.9 which shows the distance a small iron sphere (a ball bearing) drops as a function of the time since its release. The curve in Figure 1.9 was drawn by passing a smooth curve through a set of data points.

Calculating Rate of Change at a Point

Eventually we will want to be able to calculate the rate of change of a smooth function at a specific point. For example, we may want to know how fast the distance is changing (that is, we want to know the speed of the ball bearing) at, say, t = 1.5 sec. To answer this, let us back off a bit and answer a related, but different, question; namely, what is the speed of the ball bearing during the time interval from t = 1.5 sec to, say, t = 2.5 sec? This question does not mean very much without further

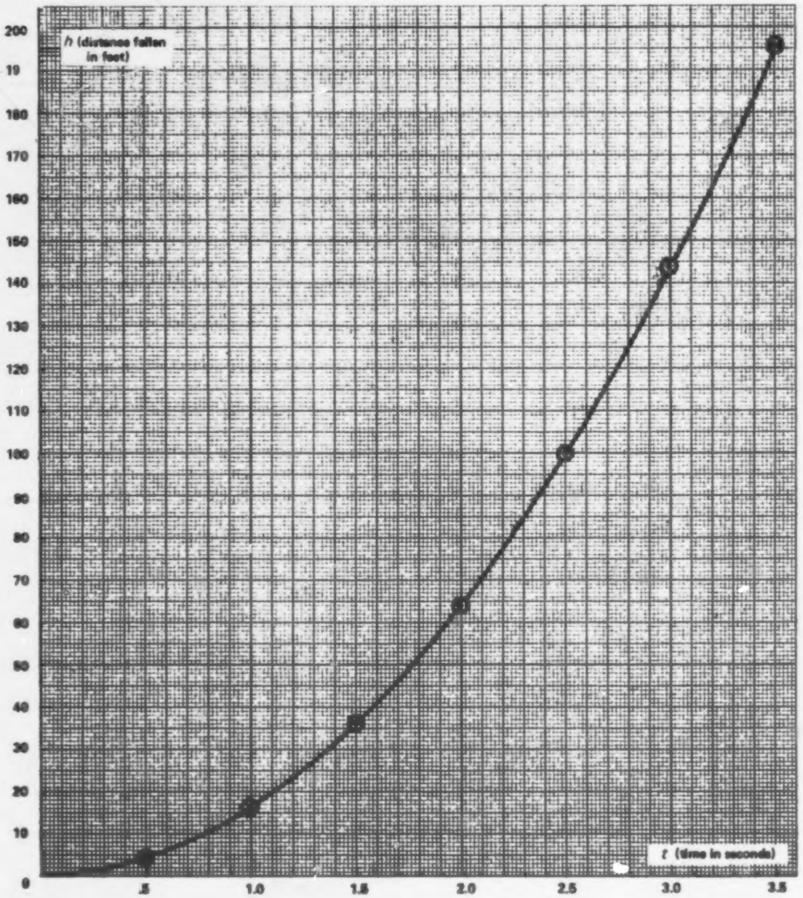


Figure 1.9. Distance fallen by a one inch diameter ball bearing versus time.

remarks. As you can see from Figure 1.9, the slope of the curve changes gradually and steadily from t = 1.5 to t = 2.5. What then do we mean when we ask what *the* speed is during this interval?

Calculating Average Speed Numerically

To avoid this problem, we define what is called the *average speed** over the interval. Figure 1.10 shows in detail the portion of Figure 1.9 from t = 1.5 sec to t = 2.5 sec. The "average" speed is obtained by finding how far the ball bearing fell during this time interval, and then dividing the distance fallen by the length of the time interval. From the graph in Figure 1.10 we see that:

when t = 1.5 sec, h = 36.0 ft
when t = 2.5 sec, h = 100 ft.

The average speed over the interval from t = 1.5 to t = 2.5 is then:

$$v_{av} = \frac{\text{distance fallen}}{\text{time to fall this distance}} = \frac{100\text{ ft} - 36\text{ ft}}{2.5\text{ sec} - 1.5\text{ sec}}$$

$$= \frac{64\text{ ft}}{1\text{ sec}} = 64\text{ ft/sec.}$$

In symbols this calculation may be written:

$$v_{av} = \frac{h_2 - h_1}{t_2 - t_1} = \frac{\Delta h}{\Delta t}$$

and is, of course, just the formula for the slope of the line from point A to point B in Figure 1.11. Point A has coordinates t_1 = 1.5 sec and h_1 = 36 ft while the point B has coordinates t_2 = 2.5 sec and h_2 = 100 ft.

---

*It is very important to realize that "average" here does not mean what it usually means. The average speed is *not* found by adding together a number of speeds and then dividing by the number of speeds. The average speed in the sense used here is that *constant* speed at which the ball bearing would have to move between t = 1.5 sec and t = 2.5 sec to cover the distance it actually does move.

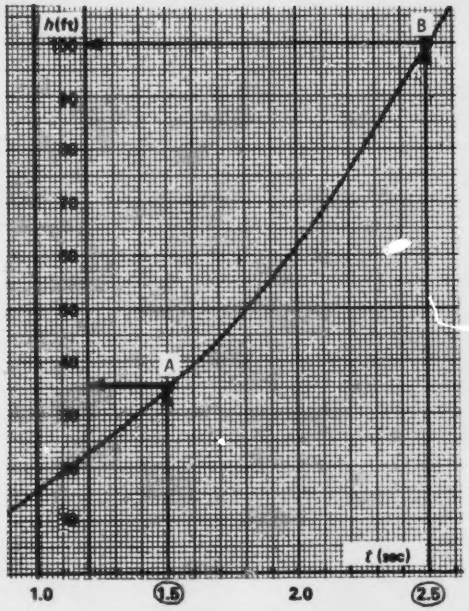


Figure 1.10. Small section of Figure 1.9.

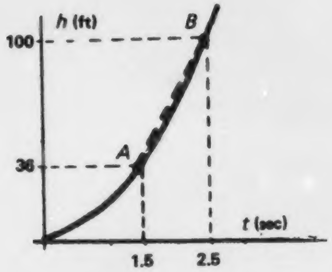


Figure 1.11. The "average" speed from t = 1.5 sec to t = 2.5 sec is equal to the slope of the line from A to B.

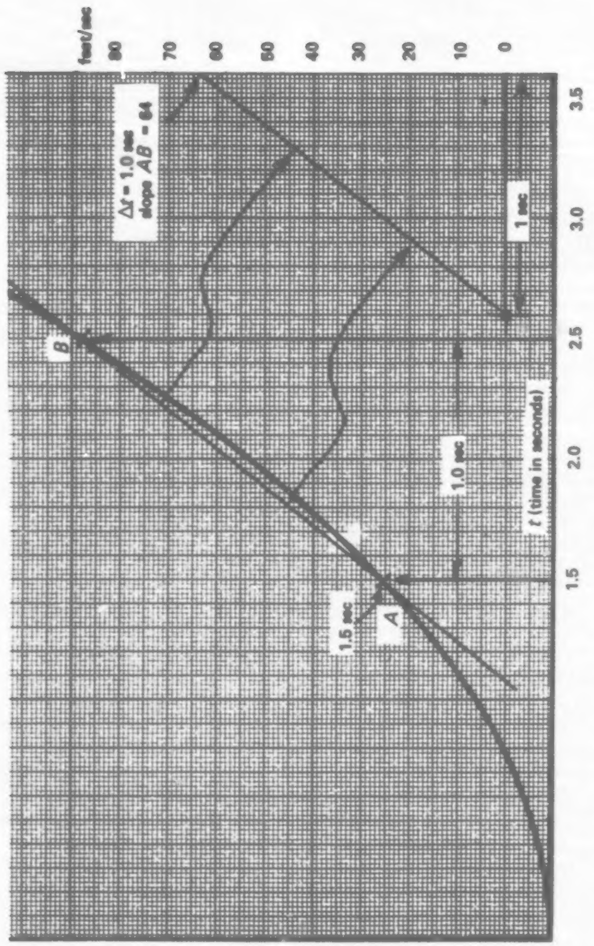


Figure 1.12. Speed from t = 1.5 sec to t = 2.5 sec by the sliding triangle method.

Calculating Average Speed Graphically

We can also find this slope (average speed) using the "sliding triangle" method described a short while earlier, which leads quickly to reasonably accurate results. In Figure 1.12 this method is used to find the average speed of the ball bearing between t = 1.5 sec and t = 2.5 sec. The result compares very favorably with the computed value of 64 ft/sec.

Objective 3: *To be able to estimate the instantaneous rate of change of a function by graphical means; that is, measure the slope of a line tangent to the curve.*

Rate of Change of a Smooth Function at a Point

When we found the average speed of a falling ball over an interval of one second, the average speed was not the actual speed at either the beginning or the end of the interval. Rather, it represented that constant speed with which the ball would have covered the 64 feet fallen in the same one second of time. Suppose now that instead of wanting the *average*, we wanted the *instantaneous* speed at the moment the clock read 1.5 seconds. The *average* speed over an interval can be measured with a tape measure and a stop watch: we measure the distance traveled and divide by the time it took to travel that distance. But obviously an *instantaneous* speed cannot be measured or calculated in the same way; we would need to measure the distance traveled in a time interval of zero length.

Taking the Average over Intervals

We will find the instantaneous speed, not by measuring a time interval of zero length, but by "sneaking up" on it: we find the average speed over shorter and shorter time intervals beginning at t = 1.5 sec. The average speed, starting at 1.5 seconds, but measured over only 0.8 seconds instead of one second, is measured in Figure 1.13 by the "sliding triangle" method and comes out to 61 feet/second.

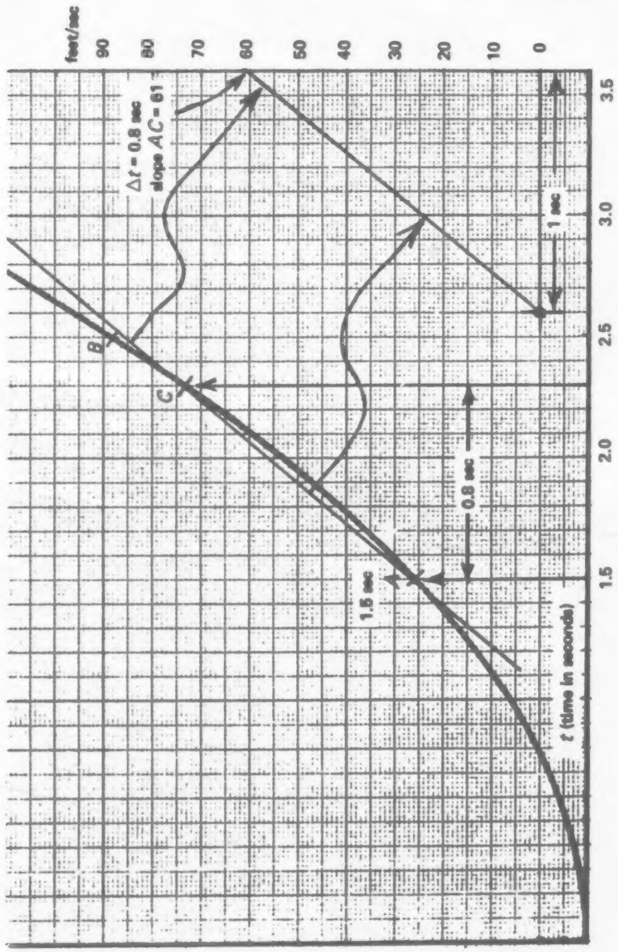


Figure 1.13. Average speed over 0.8 sec.

Figures 1.14, 1.15, and 1.16 repeat the measurement of average speed over intervals of 0.6 sec, 0.4 sec, and 0.2 sec. The results of these measurements of the average speed are shown in Table 1. We would like to continue making such graphs in order to extend the table to include even shorter time intervals. But here a practical difficulty arises: For values of Δt smaller than about 0.2 sec, it is impossible to read such graphs accurately enough to obtain reliable results. At the moment, the best we can do is to say that the instantaneous velocity at t = 1.5 sec is something near 50 ft/sec. In Appendix 2, however, we will see how this method can be used to arrive at a more accurate answer.

TABLE 1

Average Speed of Falling Ball Figured over Intervals Starting at t = 1.5 Seconds

Time Interval Δt (sec)	Average Specd v_{av} (ft/sec)
1.0	64.0
0.8	61.0
0.6	57.5
0.4	54.5
0.2	51.0

Exercise 3. Use the graph of Figure 1.17 to estimate the instantaneous speed at t = 2.0 sec. Follow the procedure described above, calculating Δh/Δt for a succession of smaller and smaller values of Δt, beginning at t = 2 sec.

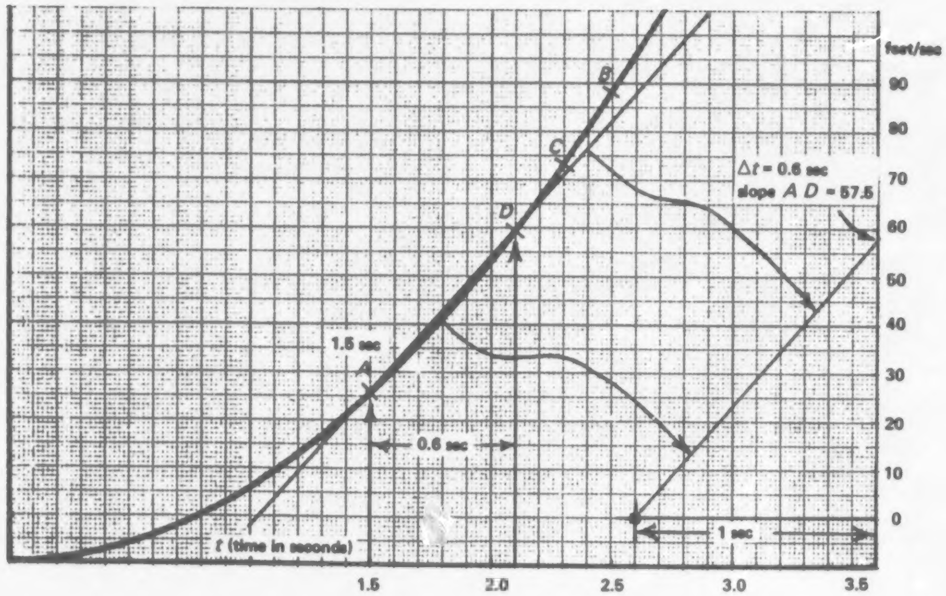


Figure 1.14. Average speed over 0.6 sec.

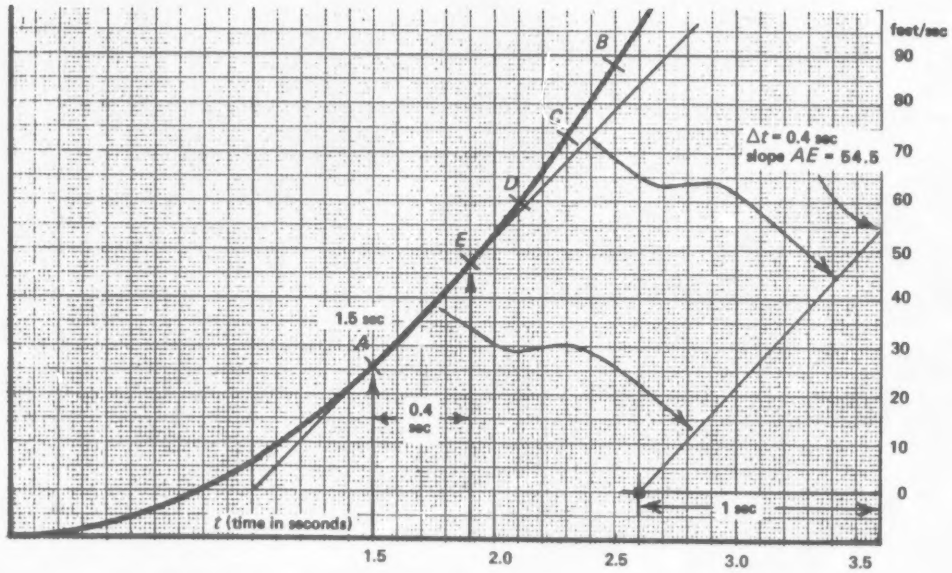


Figure 1.15. Average speed over 0.4 sec.

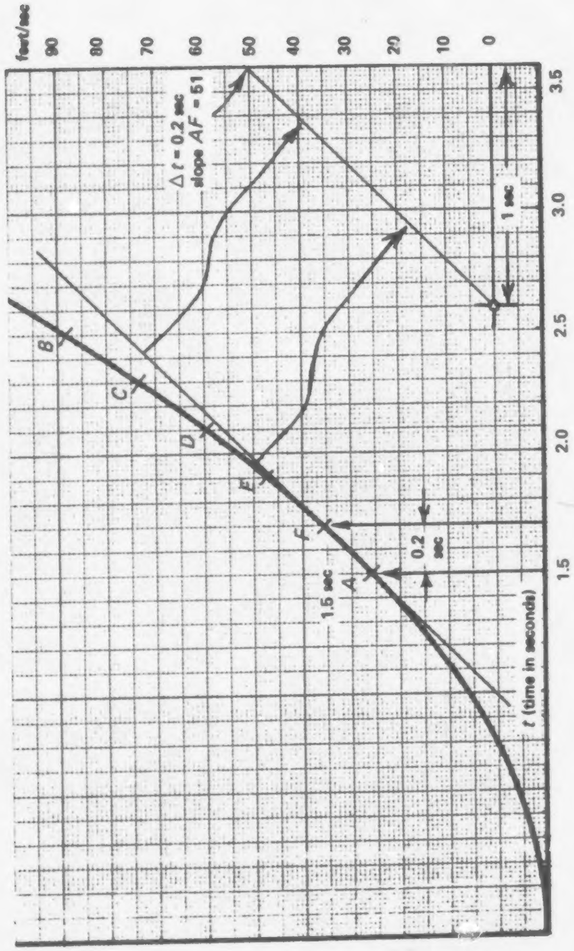


Figure 1.16. Average speed ofver 0.2 sec.

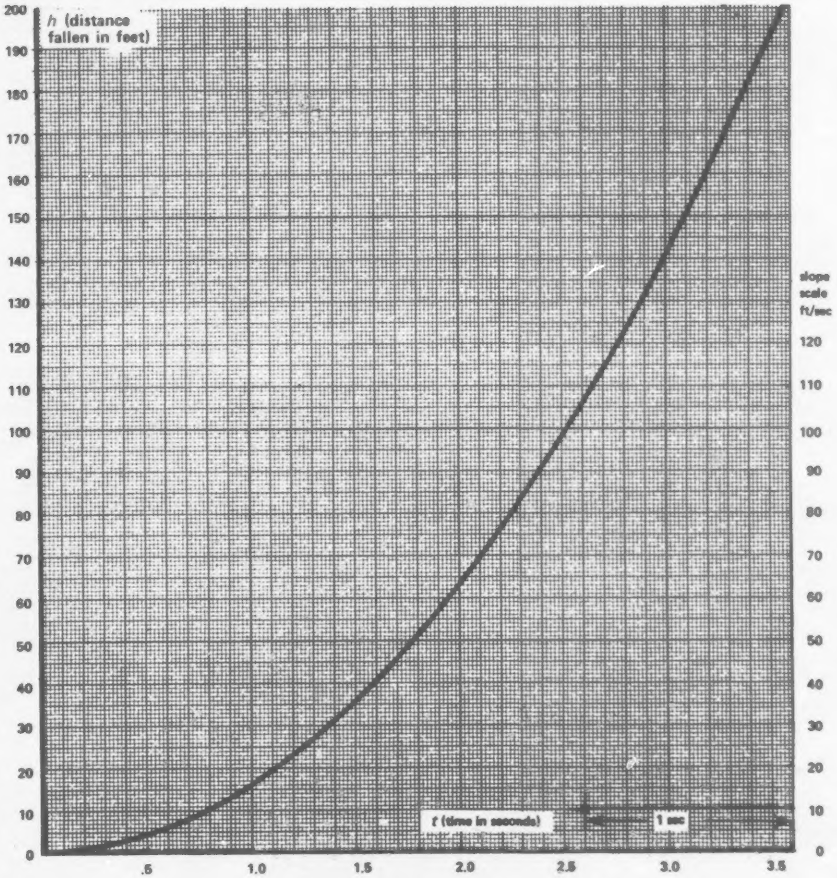


Figure 1.17. Distance fallen by a ball bearing versus time t.

Rate of Change of a Point from the Slope of a Tangent Line

If the method just used in examined in detail from a graphical point of view, it can lead to a more accurate estimate of the instantaneous velocity. Moreover, this more accurate answer can be found without the need to draw as many graphs as in Figures 1.13-1.16, or to read as many successive values of the average speed such as those in Table 1. Thus, look at the series of diagrams shown in Figure 1.18 and notice what happens to the successive lines whose slopes give the average speeds. As the time interval gets shorter and shorter, the points where the line cuts the curve move closer and closer together. As $\Delta t \to 0$ (Figure 1.18e), these two points unite to become one point; the line then touches the curve in one point only; grazing the curve. Such a line gives the slope of the curve at that point and is called a *tangent* line (from a Latin word meaning "to touch"). The slope of the tangent line is defined to the *instantaneous* speed at the point (that is, at the instant of time) where the tangent line touches the curve.

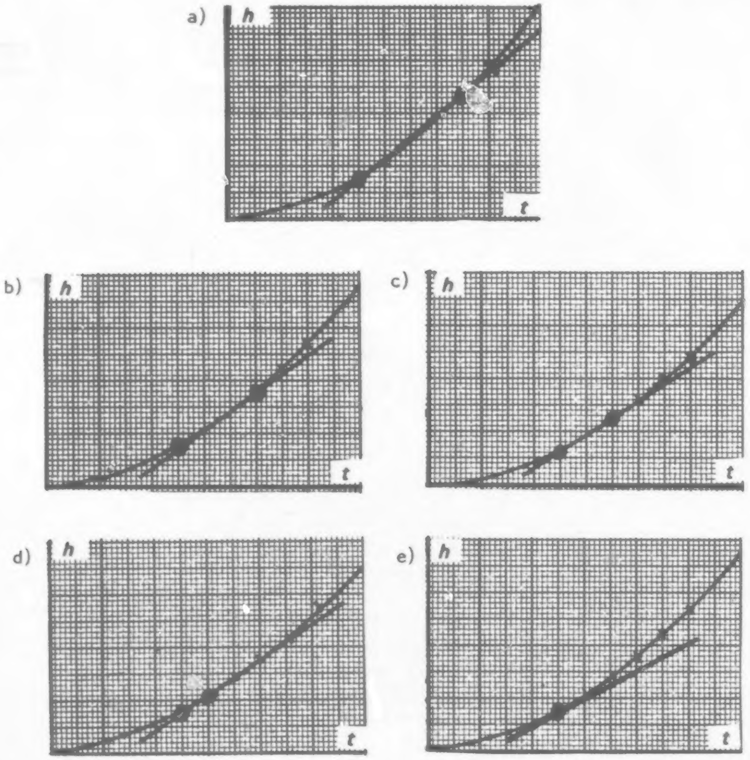


Figure 1.18. The line that cuts the curve becomes the line that touches the curve as the two points move together.

This conclusion gives us a simple way to estimate the instantaneous speed (or rate of change) from a smooth graph. We merely draw a tangent line (which can usually be done quite accurately by eye) and measure its slope. To measure the slope of the tangent line accurately, either draw a long tangent line and read off widely separated points to compute its slope as in Figure 1.19, or use the sliding triangle method as shown in Figure 1.20.

Note that the value determined from Figure 1.20 is 48 ft/sec. This, then, is our estimate of the instantaneous velocity at 1.5 sec.

Exercise 4. Estimate the instantaneous speed of the falling ball at $t = 1$, 2, and 3 sec by the tangent method. Use the graph of Figure 1.17.

Exercise 5. The graph in Figure 1.21 shows the area of an open wound versus time. In doing the following, use the smooth curve in the figure, not the dots. The healing rate is the absolute value of the rate of change of the area of the wound, measured in cm^2/day.

(a) Sketch a graph of the rate of change of the area of the wound (cm^2/day).

(b) When is the healing rate the fastest? The slowest?

(c) Use the tangent method to calculate the instantaneous rate of healing at 8 days and at 13.5 days.

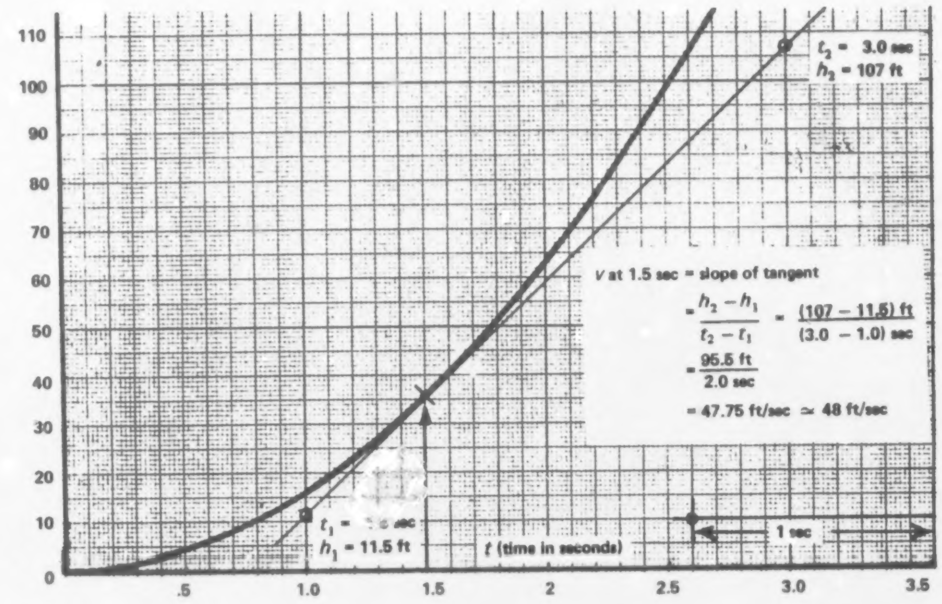


Figure 1.19. Slope of tangent from measurement of two points.

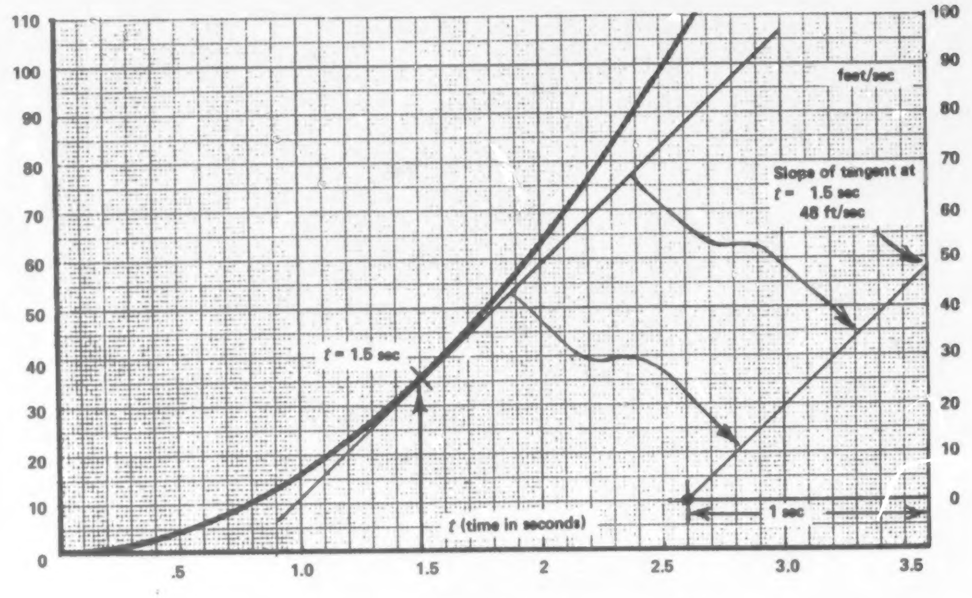


Figure 1.20. Slope of tangent by sliding triangle method.

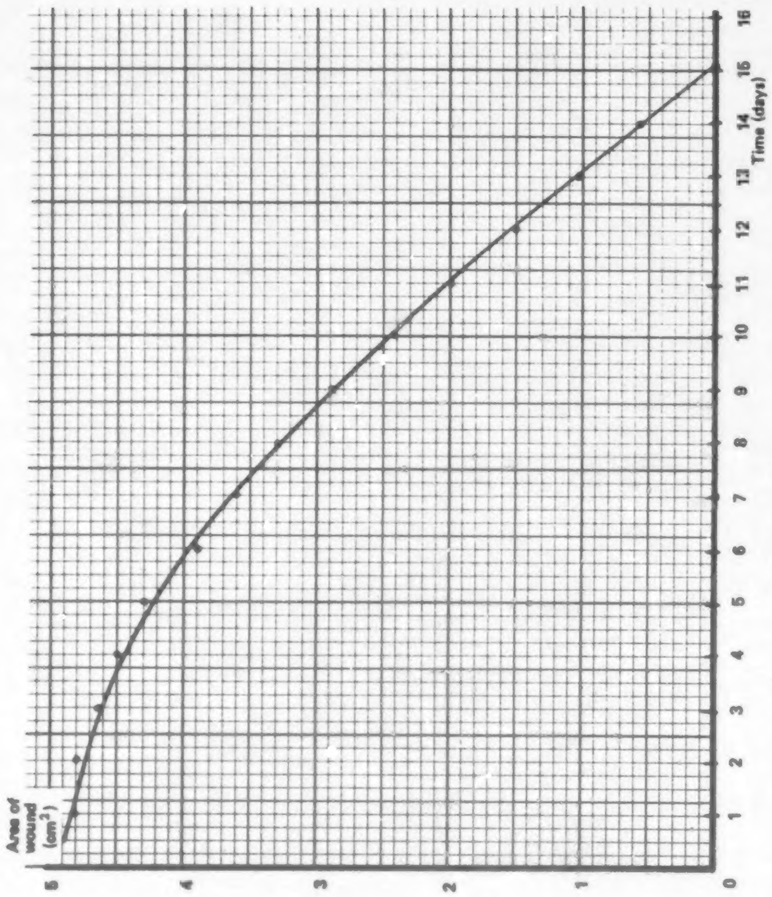


Figure 1.21. Area of an open wound versus time.

Exercise 6. Use the graph shown in Figure 1.22 to find the rate of increase in quantity of antibody 2 days, 5 days and 15 days after both the 1st and 2nd injections of antigen. Use the tangent method.

Exercise 7. Draw a pair of coordinate axes, and sketch a graph of a curve that has positive slope everywhere. Do the same for a curve whose slope is negative everywhere, and whose slope is zero everywhere.

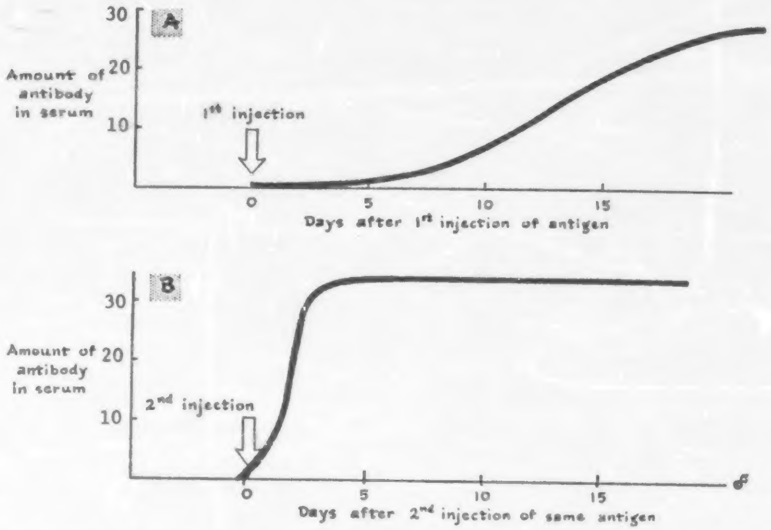


Figure 1.22. The slow production of antibody (A) after a first injection of antigen is followed by a prolonged decline in quantity of antibody (not shown). If, after this decline, a second injection of the same antigen (a "booster shot") is given, the rise in antibody level is rapid (B).

ANSWERS TO EXERCISES

1. Reno, NV and Truckee, CA: +35 ft/mile
   Green River and Evanston: +7 ft/mile
   Midsection between Omaha, NE and Julesburg, CO: +8 ft/mile
   Wells and Lovelock, NV -6 ft/mile

2. Section after Julesburg: +16 ft/mile (±2)
   Section before Green River: -5 ft/mile (±1)
   Section after Truckee: -47 ft/mile (±5)

3. The table shows that the instantaneous speed at $t = 2$ sec is about 64 ft/sec.

Δt (sec)	$\Delta h/\Delta t$ (ft/sec)
1.0	80.0
0.5	72.0
0.1	65.6
0.05	64.8
0.01	64.16
0.005	64.008

4.

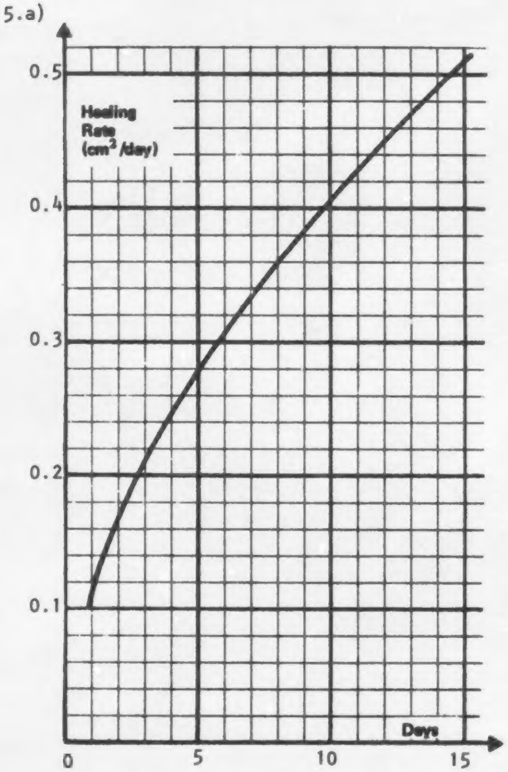
t (sec)	speed (ft/sec)
1	32
2	63
3	96

5. (a) See the graph at the top of the next page.

   (b) The healing rate is fastest at about 15 days, when the wound is nearly healed. It is slowest at the outset when the wound is newly formed.

   (c) At 8 days: about 0.4 cm^2/day.
   At 13.5 days: about 0.5 cm^2/day.

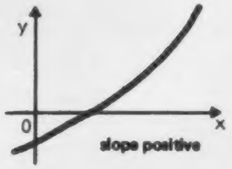
5.a)

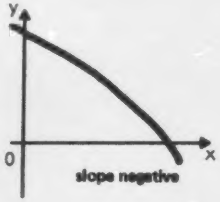


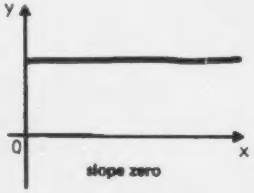
6. 1st injection: 2nd day: 0 units/day
   5th day: 0.3 units/day
   15th day: 2.2 units/day
   2nd injection: 2nd day: 15 to 20 units/day
   5th day: 0 units/day
   15th day: 0 units/day

   Note that after the 2nd injection, the amount of antibody may reach a higher level than it does after the first injection.

7. The following are examples of curves that satisfy the criteria given in the problem. Your curves may look quite different.







APPENDIX 2
RATES OF CHANGE*

Objective 1: *To be able to estimate numerically the average rate of change of a function given by a formula.*

In Appendix 1, all our information about rates of change came from graphs. We now explore how to find the average rate of change for functions given by a simple formula. The function we use is one you have probably seen before. If an object is dropped, the distance, h, which it falls in a time t is given by

$$h = 16t^2.$$

It is important to keep in mind when we use this formula that t must be in seconds and that h comes out in feet. For example, to find out how far the object has fallen at t = 1.5 sec, we calculate

$$h = 16 \times (1.5)^2 = 36 \text{ feet.}$$

Figure 2.1 shows a graph of this function.

Calculation of an Average Speed of Fall

Let us compute the average rate of change of h from $t_1 = 1.5$ sec to $t_2 = 2.5$ sec. This is really the average speed of fall and can be calculated from the formula

$$(\text{average speed}) = \frac{\Delta h}{\Delta t}$$

where Δh is the distance fallen in the time interval Δt. See Figure 2.2.

As usual, we can write

$$\Delta h = h_2 - h_1$$

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*Adapted by the UMAP Project staff from Differentiation, Second Edition, 1975, Project CALC, Education Development Center, Inc., Newton, Massachusetts, pp. 63-75.

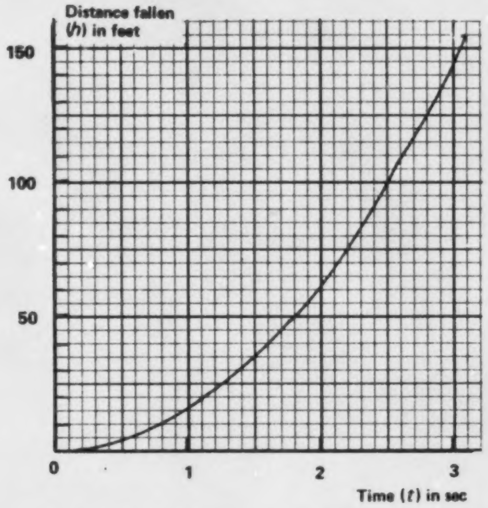


Figure 2.1. Distance as a function of time for a falling object.

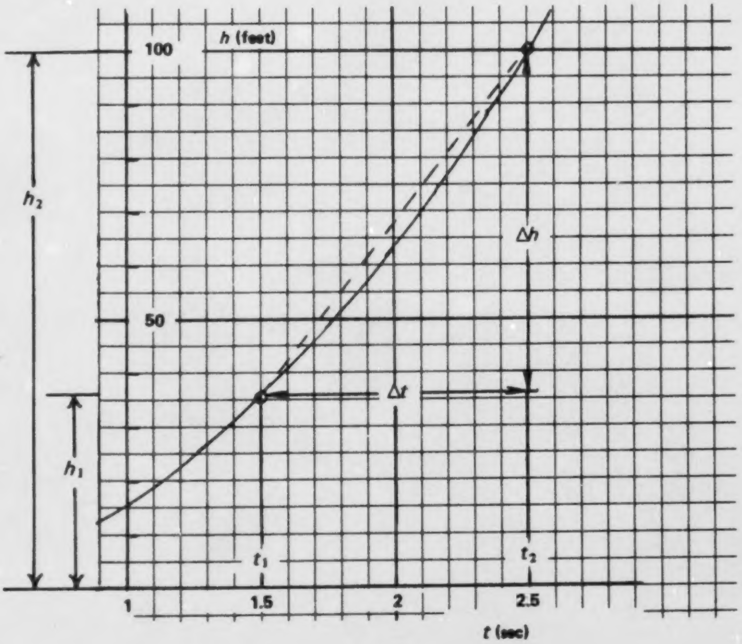


Figure 2.2. Enlarged portion of Figure 2.1.

where h_1 is the value of h when $t = t_1$ (which is 1.5 sec in this example) and h_2 is the value of h when $t = t_2$ (= 2.5 sec in this example). We also have

$$\Delta t = t_2 - t_1$$

so that

$$(\text{average speed}) = \frac{h_2 - h_1}{t_2 - t_1} .$$

We have seen equations like this in the previous unit. What is new here is that now we can calculate h_1 and h_2 from a formula instead of estimating them from a graph. Thus since $h_1 = 16t_1^2$, we have

$$h_1 = 16t_1^2 = 16(1.5)^2 = 36 \text{ feet.}$$

In just the same way,

$$h_2 = 16t_2^2 = 16(2.5)^2 = 100 \text{ feet.}$$

Accordingly,

$$(\text{average speed}) = \frac{\Delta h}{\Delta t} = \frac{h_2 - h_1}{t_2 - t_1} = \frac{64 \text{ ft}}{1 \text{ sec}} = 64 \text{ ft/sec.}$$

Objective 2: *To be able to approximate the instantaneous rate of change of a function given by a formula by taking the average rate of change over smaller and smaller intervals.*

The procedure described in the previous section is useful when we want to calculate an average speed. The result is almost certainly more precise than any we could read from a graph. But what we are *really* after is not the average speed over an interval, but the speed *at some particular instant of time* (what we have called the *instantaneous speed*). In Appendix 1, we calculated instantaneous speed by finding average speed over shorter and shorter intervals. What we did there can be summarized in the following way.

The process that defines instantaneous speed may be expressed in compact form by the following word equations:

$$\text{value approached by} \begin{bmatrix}\text{average}\\ \text{speed}\end{bmatrix} \text{ as } \Delta t \to 0 = \begin{bmatrix}\text{instantaneous}\\ \text{speed}\end{bmatrix}$$

or

$$\text{value approached by } \left[v_{av}\right] \text{ as } \Delta t \to 0 = v.$$

Numerical Approximation of Instantaneous Speed

We will again approximate the speed of a falling body at $t = 1.5$ seconds, but this time we will begin with the formula for distance of fall as a function of time, $h = 16t^2$, instead of starting with the graph as before.

Throughout the calculation, $h_1 = 36$ ft and $t_1 = 1.5$ sec. Table 1 shows the results. The first column gives the time at which the instantaneous speed is to be found, 1.5 sec. The second column gives the interval of time over which the average speed is to be calculated. As we move down the table, we let the value of Δt get smaller and smaller. Column 3 gives the value of $t_2 = t_1 + \Delta t$, which is the time at the end of the interval. Solving this simple equation for t_2 gives

$$t_2 = t_1 + \Delta t.$$

In our calculation t_1 is always equal to 1.5 sec, so we can write this last expression

$$t_2 = 1.5 + \Delta t \text{ (} t_2 \text{ and } \Delta t \text{ both in seconds).}$$

Calculation of Row 2 of Table 1

Follow the calculation of the numbers in Table 1 by examining in detail the second row (which is typical of all rows in the table). Column 1 of this row is, of course, $t_1 = 1.5$ sec. In Column 2 of the second row we have the number 0.80 sec. This means that the second row corresponds to the choice $\Delta t = 0.80$ sec. Column 3 gives the value of t_2 which is gotten from the last equation above: putting $\Delta t = 0.80$ we find $t_2 = 1.5 + 0.80 = 2.30$, the number given in the second row of Column 3 of the table.

TABLE 1

Calculation of Average Speed
Over Shorter and Shorter Time Intervals

(1)		(2)		(3)	(4)	(5)	(6)
Time at Beginning of Interval (sec)	+	Length of Time Interval (sec)	=	Time at End of Interval (sec)	Height at End of Interval (ft)	Change in Height (ft)	Average Speed Over Interval (ft/sec)
t_1	+	Δt	=	t_2	h_2	Δh	$\Delta h/\Delta t$
1.5	+	1.00	=	2.50	100	64	64
1.5	+	0.80	=	2.30	84.64	48.64	60.8
1.5	+	0.60	=	2.10	70.56	34.56	57.6
1.5	+	0.40	=	____	____	____	____
1.5	+	0.20	=	1.70	46.24	10.24	51.2
1.5	+	0.10	=	1.60	40.96	4.96	49.6
1.5	+	0.01	=	1.51	36.4816	0.4816	48.16
1.5	+	0.001	=	1.501	36.04802	0.04802	48.02

Column 4 of Table 1 is h_2, the distance the object has fallen by $t_2 = 2.3$ sec. This value is obtained from our formula for h in terms of t: $h = 16t^2$. Thus when $t = t_2 = 2.3$ sec, we get

$$h = h_2 = 16(2.3)^2 = 84.64 \text{ ft}$$

and you can see this number in the second row of Column 4. These numbers, as well as those we will calculate next, are shown in Figure 2.3, which is exactly like Figure 2.2 except that it corresponds to $\Delta t = 0.80$ sec rather than 1 sec.

Column 5 gives h which, by definition is

$$\Delta h = h_2 - h_1$$

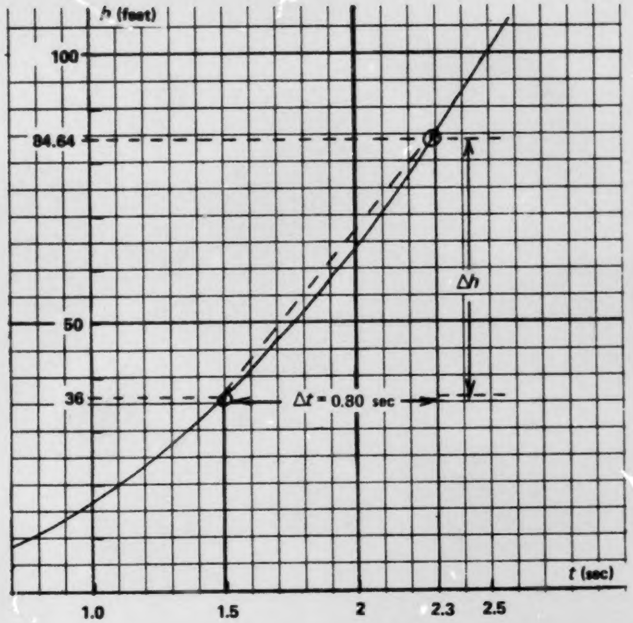


Figure 2.3

Since h_1 = 36 feet throughout our calculation this equation becomes

$$\Delta h = h_2 - 36 \text{ (} h_2 \text{ and } \Delta h \text{ both in feet).}$$

From the previous calculation (Column 3) we know that h_2 = 84.64 ft. It follows then that for row 2 of Table 1

$$\Delta h = 84.64 - 36 = 48.64 \text{ ft}$$

and this result appears in Column 5.

The last column of the table (Column 6) is the average speed of the falling object over an interval of time Δt = 0.80 sec beginning at $t = t_1$ = 1.5 sec. Using Δt = 0.80 sec (Column 2) and Δh = 48.64 ft (Column 5) we get

$$\frac{\Delta h}{\Delta t} \quad \frac{48.64 \text{ ft}}{0.80 \text{ sec}} = 60.8 \text{ ft/sec},$$

the number listed in Column 6 of Table 1.

All of Table 1 is made this way, each value of Δt (Column 2) being chosen smaller than the value in the previous row. Spot check a few of the numbers in other rows of Table 1 to be sure you understand how each is calculated.

Exercise 1. The fourth row of Table 1 (corresponding to Δt = 0.40 sec) has been left incomplete. Follow the procedure discussed above and fill in the row. Consult Table 2 to see if the value you obtain for $\Delta h/\Delta t$ is correct.

Results of the Calculation as $\Delta t \to 0$

Now that we have Table 1, we can make a second, much simpler table, by omitting everything except the Δt and $\Delta h/\Delta t$ columns, as is done in Table 2. Since we are interested in the average speed, $\Delta h/\Delta t$, for smaller and smaller values of Δt, these are the important columns of Table 1; the other columns were put in only to assist us in making the calculations.

TABLE 2

The Average Speed
Over Shorter and Shorter Time Intervals

Length of Time Interval (sec) Δt	Average Speed Over Interval (ft/sec) $\Delta h/\Delta t$
1.00	64
0.80	60.8
0.60	57.6
0.40	54.4
0.20	51.2
0.10	49.6
0.01	48.16
0.001	48.02

The values of average speed from Table 2 are plotted in Figure 2.4. By looking at either the table or the graph it is possible to see that, as Δt grows smaller and approahces 0, $\Delta h/\Delta t$ gets closer and closer to a value at or near 48 ft/sec. In a word equation:

$$\text{value approached by } \frac{\Delta h}{\Delta t} \text{ as } \Delta t \to 0 = 48 \text{ ft/sec.}$$

Since we already have stated that

$$\text{value approached by } \frac{\Delta h}{\Delta t} \text{ as } \Delta t \to 0 = \begin{bmatrix}\text{instantaneous}\\ \text{speed}\end{bmatrix} = v,$$

the value

$$v = 48 \text{ ft/sec}$$

is our approximation for the instantaneous speed of fall at $t = 1.5$ sec.

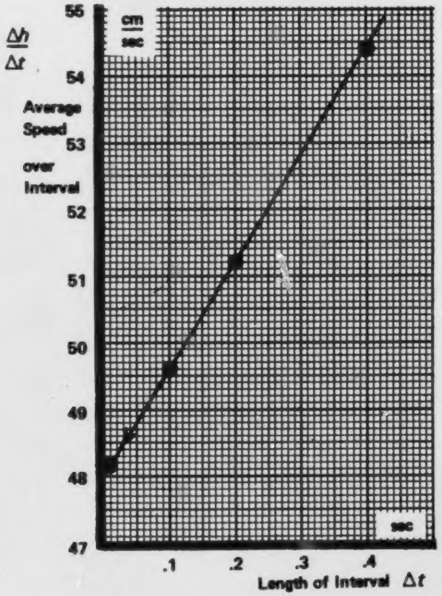


Figure 2.4. Some values of Table 2 plotted to see what happens to $\frac{\Delta h}{\Delta t}$ as $\Delta t \to 0$.

Exercise 2. Compare the results of the calculations shown in Table 2 with your calculations of average speed using the tangent line done in Table 1 of Appendix 1.

Exercise 3. Use the formula $h = 16t^2$ and the numerical method just described to find (at least approximately) the speed of the falling object at $t = 2$ sec.

Exercise 4 (Calculator Exercise). If you have a calculator at your disposal, redo the calculation of Exercise 3 to find the speed of the falling object at $t = 0, 0.5, 1, 1.5, 2, 2.5, 3$ sec, etc. Plot the speed as a function of t. What kind of curve is your graph?

Another Example: Calculation of a Slope at a Point

Let us try to find the *slope* of the graph of the function $y = x^3$ at $x = 2$. You will see that although in the previous example we calculated an *instantaneous velocity*, and in this second example we will calculate a *slope*, and the procedures involved are identical.

In the last example, we calculated *instantaneous velocity* from average velocity. In this example, we calculate the *instantaneous rate of change* of a function (the slope of its graph at a point) from the average rate of change. Definitions and procedures are identical and may be expressed in the following form:

$$\text{value approached by } \begin{bmatrix}\text{average}\\ \text{rate of}\\ \text{change}\end{bmatrix} \text{ as } \Delta x \to 0 = \begin{bmatrix}\text{instantaneous}\\ \text{rate of change}\end{bmatrix},$$

or

$$\text{value approached by } \begin{bmatrix}\frac{\Delta y}{\Delta x}\end{bmatrix} \text{ as } \Delta x \to 0 = \begin{matrix}\text{slope of curve}\\ \text{at a point}\end{matrix}.$$

Figure 2.5 is a graph of the function $y = x^3$. You should make a few calculations yourself, and verify that the curve in Figure 2.5 really is a graph of $y = x^3$. (Figure 2.5 shows $y = x^3$ only for *positive* values of x. Can you draw a graph of $y = x^3$ for *negative* values of x, say from 0 to -5?)

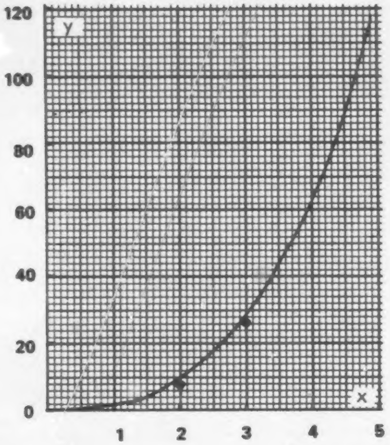


Figure 2.5. Graph of $y = x^3$.

Numerical Calculation of the Slope at x = 2

To find the slope of the curve in Figure 2.5 at $x = 2$, we proceed as we did above in calculating instantaneous velocity. That is, we calculate

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

for smaller and smaller values of Δx (where $\Delta x = x_2 - x_1$). In our example, $x_1 = 2$ and so $y_1 = 2^3 = 8$. Hence,

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{\Delta x} = \frac{y_2 - 8}{\Delta x} .$$

The results of such a calculation are shown in Table 3. This table is made exactly the way we made Table 1, so we will not discuss it in detail, but you should spot check a few of the numbers. (We have rounded off the numbers in Table 3 so as to keep no more than two figures after the decimal point in $\Delta y/\Delta x$.)

We now have a method for finding instantaneous rates of change. Speed, or more correctly instantaneous speed, is an example of this. Practice this method on the next few exercises.

TABLE 3

Approximating the Slope of $y = x^3$
at $x = 2$ by Letting Δx Approach Zero

x_1	+	Δx	=	x_2	y_2	$\Delta y = y_2 - 8$	$\Delta y/\Delta x$
2	+	1.0	=	3.0	27.	19.0	19
2	+	0.50	=	2.50	15.625	7.625	15.25
2	+	0.20	=	2.20	10.648	2.648	13.24
2	+	0.10	=	2.10	9.261	1.261	12.61
2	+	0.05	=	2.05	8.6151	0.615	12.30
2	+	0.02	=	2.02	8.2424	0.2424	12.12
2	+	0.01	=	____	____	____	____
2	+	0.005	=	____	____	____	____

Exercise 5. Complete the last two rows of Table 3 to verify that $\Delta y/\Delta x$ approaches 12 as Δx approaches zero.

Exercise 6. Verify the answer of Exercise 5 by drawing a line tangent to the curve in Figure 2.5 at $x = 2$, and measuring its slope.

Exercise 7. Adapt the method used above to find the slope of $y = x^2$ at $x = 1$.

Exercise 8 (Calculator Exercise). Use the methods of this unit to find the slope of $y = x^2$ at $x = 0, 0.5, 1, 1.5, 2, 2.5,$ and 3 sec. Graph the slope versus x. What kind of curve do you get?

Some Further Comments

It may have seemed that we found both instantaneous speed and the slope at a point in a rather roundabout way. In each case we calculated an *average* rate of change ($\Delta h/\Delta t$ or $\Delta y/\Delta x$) over an interval (Δt or Δx) which got closer and and closer to zero. But why be so sneaky? Why not just set Δt or Δx equal to zero instead of letting it *approach* zero?

To answer this question, consider the kind of measurements we make in order to get the numbers displayed in Table 1. Put in its simplest terms, we determine the distance, Δh, which an object falls in a time Δt. We then calculate Δh/Δt. Putting Δt = 0 directly would mean that we would have to determine the distance the object falls in a time "interval" of zero length. But in zero time the object moves zero distance so that corresponding to Δt = 0 we would have Δh = 0, and the ratio Δh/Δt = 0/0. The additional row for Table 1 which would correspond to this calculation is shown in Table 4.

TABLE 4

An Additional Row for Table 1
Corresponding to Δt = 0

t_1(sec) + t(sec) = t_2(sec)	h_2(ft)	Δh(ft)	Δh/Δt(ft/sec)
1.5 + 0 = 1.50	36	0	$\frac{0}{0}$

There are two things wrong with the "result" 0/0. First, it is the same result we would get for the object whatever its true speed might be. Whether it is moving so fast that all we see is a blur as it passes, or so slowly that it is at the proverbial "snail's pace," our "answer" will be 0/0. Obviously a quantity which does not distinguish between something moving rapidly and something moving slowly can hardly be used as the definition of instantaneous velocity.

The second difficulty with 0/0 is a mathematical one. 0/0 is, mathematically speaking, a meaningless symbol; it is impossible to ascribe a unique numerical value to it.

All this helps to explain why we must "sneak up" on the instantaneous velocity by determining the value Δh/Δt *approaches* as Δt *approaches* zero. We can summarize our discussion of instantaneous velocity by saying that

$$\frac{\Delta h}{\Delta t} \rightarrow v \quad \text{as } \Delta t \rightarrow 0.$$

Here v is the instantaneous velocity at some specific time, Δt is an interval which begins at that time, and Δh is the distance the object moves during that interval.

ANSWERS TO EXERCISES

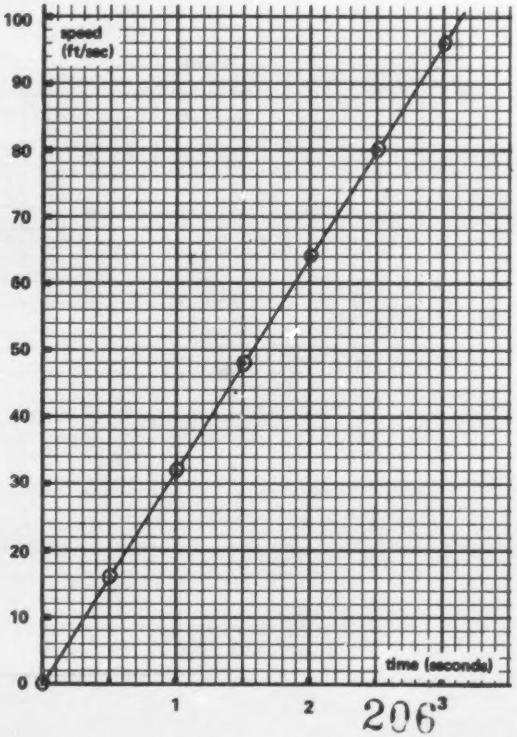
1.

t_1	+	Δt	=	t_2	h_2	Δh	Δh/Δt
1.5	+	0.40	=	1.90	57.76	21.76	54.4

2. The two results should agree except for small errors in finding slopes by the tangent line method.

3. At t = 2 sec, v = 64 ft/sec.

4. The graph is a straight line.



5.

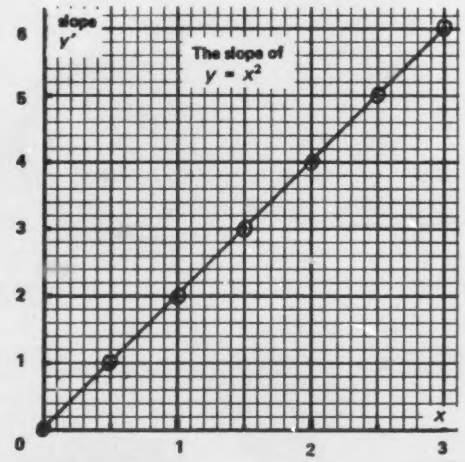
x_1	+	Δx	=	x_2	y_2	Δy	$\Delta y/\Delta x$
2	+	0.01	=	2.01	8.120601	0.120601	12.0601
2	+	0.005	=	2.005	8.0601501	0.0601501	12.03002

6. The two results should agree except for small errors in finding slopes by the tangent line method.

7.

x_1	+	Δx	=	x_2	y_2	$\Delta y = y_2 - 1$	$\Delta y/\Delta x$
1	+	0.1	=	1.1	1.21	0.21	2.1
1	+	0.01	=	1.01	1.0201	0.0201	2.01
1	+	0.001	=	1.001	1.002001	0.002001	2.001

8. The graph is a straight line.



APPENDIX 3

Differentiation Formulas from Calculus

If u and v are differentiable functions of x, then the sum and product functions, u + v and u · v, are differentiable functions of x, and their derivatives are given by the formulas

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{(Sum Rule)}$$

and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + u\frac{du}{dx}. \qquad \text{(Product Rule)}$$

Further, if $v(x) \neq 0$, then the quotient function u/v is differentiable at x, and its derivative is given by the formula

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}. \qquad \text{(Quotient Rule)}$$

Finally, if y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \qquad \text{(Chain Rule)}$$

APPENDIX 4

The formulas for the derivatives of the trigonometric functions of x are listed below.

$\frac{d}{dx}\sin x = \cos x.$

$\frac{d}{dx}\cos x = -\sin x.$

$\frac{d}{dx}\tan x = \sec^2 x.$

$\frac{d}{dx}\cot x = -\csc^2 x.$

$\frac{d}{dx}\sec x = \sec x \tan x.$

$\frac{d}{dx}\csc x = -\csc x \cot x.$

The formulas for the derivatives with respect to x for trigonometric functions of u where u is a differentiable function of x are listed below.

$\frac{d}{dx}\sin u = \cos u \frac{du}{dx}.$

$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}.$

$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}.$

$\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}.$

$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}.$

$\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}.$

V. ANSWERS TO MODEL EXAMS

Answers to Model Exam from Unit 158

1. Calculus is not needed. The answer is $h = 1200 \tan 0.3 \approx 371.2$ meters.

2. Calculus is needed. The first and second derivatives of voltage function will be found and the second derivative test will be used. The maximum voltage is $v = 1$, when $t = \frac{\pi}{4}$.

3. Calculus is not needed. The formula for the area of a sector will give approximately 51 in.^{2} .

4. Calculus is needed. The area of the triangle, $A = 18 \sin\theta \cos\theta$, is obtained by right triangle trigonometry. Then, as in problem 2, the second derivative test will be used. The answer is $\theta = \frac{\pi}{4}$.

5. A problem similar to any of the four problems given in the unit or in Problems 2 and 4 of this exam would be an acceptable answer.

Answers to Model Exam from Unit 159

1. a. $\frac{d}{dx}(\sin x) = \cos x$

   b. $\frac{d}{dx}(\cos x) = -\sin x$

2. radian.

3. Yes. The result will not hold if degree measure is used.

4. a. -0.7

   b. -0.9

5.

Δx	$\Delta y = \sin(0.4 + \Delta x) - \sin 0.4$	$\Delta y/\Delta x$
.1	.0900	.900
.01	.00919	.919
.001	.000921	.921
.0001	.000921	.921
-.1	-.0939	.939
-.01	-.00923	.923
-.001	-.000921	.921
-.0001	-.0000921	.921

The value of the derivative $y = \sin x$ at $x = 0.4$ is approximately 0.921.

Answers to Model Exam from Unit 160

1. $\frac{\sin h}{h}$ is sandwiched between two other expressions, each of which has a limit of 1 as $h \to 0$.
2. $\lim\limits_{h\to 0} \frac{\cos h - 1}{h} = 0$.
3. $\cos\left(\frac{17\pi}{180}\right)$.
4. $-\sin\left(\frac{39\pi}{180}\right)$.
5. $2x \cos(x^2 - 3)$.
6. $2 \cos x - 3 \sin 3x$.
7. $2 \sin x \cos x$.
8. $-\sin^2 x + \cos^2 x$.
9. $-(6x - 1) \sin(3x^2 - x)$
10. $\cos^3 x - 2\sin^2 x \cos x$
11. $-\cos x + c$.
12. $\frac{1}{3} \sin 3x + c$.
13. $5x + 2\cos x + c$.
14. $\frac{dF}{d\theta} = \frac{-KW(K\cos\theta - \sin\theta)}{(K\sin\theta + \cos\theta)^2}$.

Answers to Model Exam from Unit 161

1. $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{\cos^2 x}$

   $= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$.

2. $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2 x}$

   $= \frac{-\cos x}{\sin^2 x} = \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$.

3. $\frac{dy}{dx} = -\csc^2 x$.
4. $\frac{dy}{dx} = \sec x \tan x$.
5. $\frac{dy}{dx} = 3 \sec(3x + 5)$.
6. $\frac{dy}{dx} = 2 \cos 2x + 2x \sec^2 x^2$.
7. $\frac{dy}{dx} = x \sec^2 x + \tan x$.
8. $\frac{dy}{dx} = \frac{-\sin x\,(1 + \tan 2x) - 2 \cos x \tan 2x}{(1 + \tan 2x)^2}$.

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STUDENT FORM 1

Request for Help

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name______________________________ Unit No.________

Page________
○ Upper
○ Middle
○ Lower

OR

Section__________
Paragraph________

OR

Model Exam
Problem No.______
Text
Problem No.______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature________________________

Please use reverse if necessary.

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STUDENT FORM 2

Unit Questionnaire

Name________________________________Unit No.____________Date____________

Institution______________________________Course No.________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much	Somewhat	About	Somewhat	Much
____Longer	____Longer	____the Same	____Shorter	____Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain____

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain____

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

UMAP

**MODULES AND
MONOGRAPHS IN
UNDERGRADUATE
MATHEMATICS
AND ITS
APPLICATIONS**

Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ
α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ σ τ υ φ χ ψ ω α ϑ
Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ
α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ σ τ υ φ χ ψ ω α ϑ
Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ
α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ σ τ υ φ χ ψ ω α ϑ

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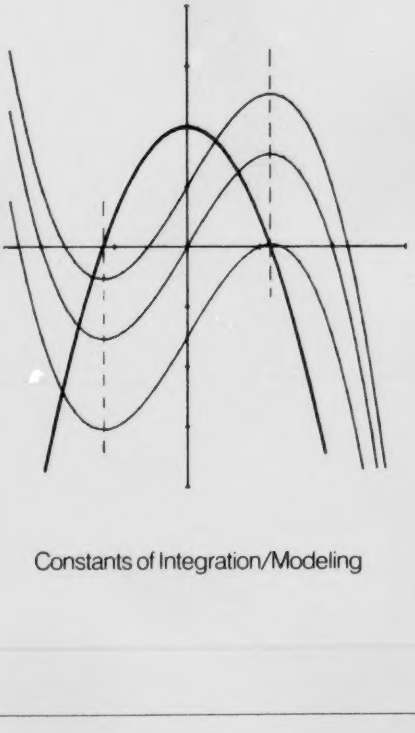
MODULE 162

Determining Constants of Integration

by Ross L. Finney



Constants of Integration/Modeling

SE 036 477

Intermodular Description Sheet: *UMAP Unit 162*

Title: DETERMINING CONSTANTS OF INTEGRATION

Author: Ross L. Finney
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

Review Stage/Date: IV 8/30/80

Classification: CONSTANTS OF INT/MODELING

Prerequisite Skills:
1. Differentiate and graph polynomials.
2. Know that if the derivative of a function is 0 on an interval, then the function is a constant on that interval.
3. Ability to differentiate kx, x^r, ln x, and e^x, and to graph these functions.

Output Skills:
1. Know that an indefinite integral is a family of functions with a common derivative.
2. Be able to write the indefinite integral of a linear combination of functions whose indefinite integrals are known.
3. Be able to determine from an indefinite integral the particular function that satisfies a given initial condition.
4. Be able to determine constants of integration from initial conditions stated in various ways.

Related Units:
Developing the Fundamental Theorem of Calculus *(Unit 323)*

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DETERMINING CONSTANTS OF INTEGRATION

by

Ross L. Finney
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

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10. ANSWERS TO EXERCISES 26

DETERMINING CONSTANTS OF INTEGRATION

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1. INTRODUCTION

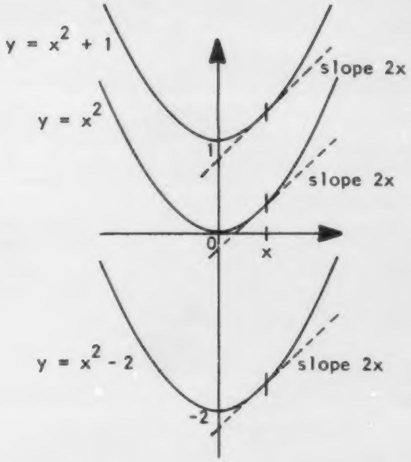
As you know, there are times when we have information about the derivative of a function and wish to conclude from it information about the function itself.

DERIVATIVE	FUNCTION
Velocity	Distance
Acceleration	Velocity
Marginal cost	Cost
Rate of growth of a population	Size of the population

The reason that we can often succeed in determining functions from their derivatives is that whenever two functions have the same derivative on an interval the functions differ only by a constant on the interval. Thus, if we can find even one function that has the given derivative, we know that the function we seek cannot differ from it by more than a constant. The basic fact is this:

IF $f'(x) = g'(x)$ for ALL VALUES OF x
IN SOME INTERVAL, THEN FOR SOME CONSTANT C
$f(x) - g(x) = C$ or $f(x) = g(x) + C$
FOR ALL VALUES OF x IN THE INTERVAL.

For every value of x, the two functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 2$ have the derivative $f'(x) = g'(x) = 2x$. Notice that $f(x) = g(x) + 3$ for all x. The value of C in the rule stated above is $C = 3$.



To get the graph of f, we may slide the graph of $y = x^2$ up 1 unit. To get the graph of g, we slide it down 2 units. The three graphs have the same slope at any x.

Functions whose derivatives are equal only at isolated points, however, do not have to differ by a constant.

EXAMPLE 1. The difference of the functions $f(x) = 2x^2$ and $g(x) = x^2$ is $2x^2 - x^2 = x^2$, and not a constant. However, the derivatives of these two functions have the same value at $x = 0$, as you can see in the following table.

THE FUNCTIONS	THEIR DERIVATIVES	THEIR DERIVATIVES AT $x = 0$
$f(x) = 2x^2$	$f'(x) = 4x$	0
$g(x) = x^2$	$g'(x) = 2x$	0

Exercises

1. Find two values of x at which the derivatives of $f(x) = 2x^3$ and $g(x) = 3x^2$ are equal.

2. Suppose that $f(x)$ and $g(x)$ are two functions that have derivatives on some interval, and that

$$f(x) - g(x) = C$$

on the interval.

   a) Differentiate both sides of the preceding equation to show that differentiable functions that differ by a constant on an interval have the same derivatives on the interval.

   b) Show that

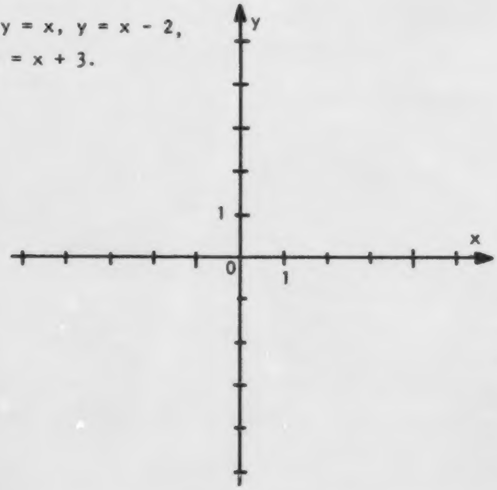
$$2x^3 - 3x^2$$

   is not constant on any interval.

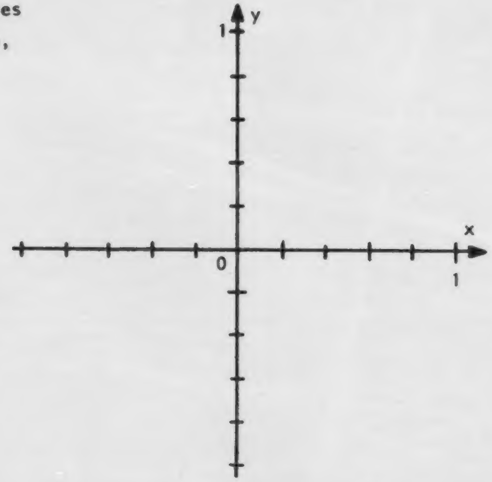
3. Find two more functions whose difference is not a constant but whose derivatives agree at one or more points.

In Exercises 4 and 5, use the coordinate axes provided to graph the given functions.

4. Graph the lines $y = x$, $y = x - 2$, $y = x + 1$, and $y = x + 3$.



5. Graph the cubic curves $y = x^3$, $y = x^3 - 0.6$, and $y = x^3 + 0.4$.



2. INDEFINITE INTEGRALS

Since the derivative of $5x^2$ is $10x$, any function $f(x)$ that has the derivative

$$f'(x) = 10x$$

must have the form

$$f(x) = 5x^2 + C$$

for some constant C. Without more information we cannot learn the value of C, but at least we have *determined* f *up to a constant*, as we say. We call the family of functions $5x^2 + C$ the *indefinite integral* of $10x$, and we show this by writing

$$\int 10x\,dx = 5x^2 + C.$$

The constant C in this formula is called the *constant of integration*.

Many indefinite integrals may be found by reversing derivative formulas we already know. Here are some examples.

	DERIVATIVE FORMULA		COMPANION INTEGRAL FORMULA
1.	$\frac{d}{dx}(kx) = k$	1'.	$\int k\ dx = kx + C$
2.	$\frac{d}{dx}(x^r) = rx^{r-1}$	2'.	$\int rx^{r-1}\ dx = x^r + C$

If we change all the r's in formulas (2) and (2') to r + 1, and then divide both sides of the formulas so obtained by (r + 1) we get formulas (3) and (3') shown below. Formula (3') tends to be more useful than formula (2').

3.	$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = x^r$	3'.	$\int x^r\ dx = \frac{x^{r+1}}{r+1} + C$

Formulas (3) and (3') don't work when r = -1, but the next formula takes care of this case.

4.	$\frac{d}{dx}(\ln \lvert x \rvert) = \frac{1}{x}$	4'.	$\int \frac{1}{x}\ dx = \ln \lvert x \rvert + C$
5.	$\frac{d}{dx}(e^x) = e^x$	5'.	$\int e^x\ dx = e^x + C$

Notice how nice a function e^x is!

EXAMPLE 2.

$$\int -5\ dx = -5x + C$$
$$\int 4x^3\ dx = x^4 + C$$
$$\int x^3\ dx = \frac{x^4}{4} + C$$

EXERCISES

Complete the equations in Exercises 6 - 21.

6. $\int 4\ dx = \underline{\qquad\qquad}$

7. $\int -25\ dx = \underline{\qquad\qquad}$

8. $\int \underline{\qquad\qquad} dx = 17x + C$

9. $\int \underline{\qquad\qquad} dx = -3x + C$

10. $\int x^2 dx = \underline{\qquad\qquad}$

11. $\int x^5 dx = \underline{\qquad\qquad}$

12. $\int \frac{dx}{x} = \ln|x| + \underline{\qquad\qquad}$

13. $\int \frac{1}{x}\, dx = \underline{\qquad\qquad}$

14. $\int \underline{\qquad\qquad} dx = \frac{x^8}{8} + C$

15. $\int \frac{x^3}{3}\, dx = \underline{\qquad\qquad}$

16. $\int 103x^{102}\, dx = \underline{\qquad\qquad}$

17. $\int \underline{\qquad\qquad} dx = \frac{x^{17}}{17} + C$

18. $\int \underline{\qquad\qquad} dx = e^x + C$

19. $\int \underline{\qquad\qquad} dx = x^4 + C$

20. $\int \underline{\qquad\qquad} dx = \frac{x^4}{4} + C$

21. $\int \underline{\qquad\qquad} dx = -103.5x + C$

In Exercises 22-25, the letters a, b, k and m are constants. Complete each formula.

22. $\int k\, dx = \underline{\qquad\qquad}$

23. $\int a\, dx = \underline{\qquad\qquad}$

24. $\int \underline{\qquad\qquad} dx = mx + C$

25. $\int \underline{\qquad\qquad} dx = -bx + C$

So far we have used x as the only variable of integration, but other letters are commonly used. Complete the equations in Exercises 26-33.

26. $\int 32\, dt = \underline{\qquad\qquad}$

27. $\int \underline{\qquad\qquad} dt = at + C$

28. $\int P^2\, dP = \underline{\qquad\qquad}$

29. $\int \underline{\qquad\qquad} dB = B^3 + C$

30. $\int 5s^4\, ds = \underline{\qquad\qquad}$

31. $\int v\, dv = \underline{\qquad\qquad}$

32. $\int e^y\, dy = \underline{\qquad\qquad}$

33. $\int \underline{\qquad\qquad} dR = \ln|R| + C$

In Exercises 34-39, the letters with subscripts are constants. Complete each formula.

34. $\int t_0\, dt = \underline{\qquad\qquad}$

35. $\int \underline{\qquad\qquad} dt = 32t_0 t + C$

36. $\int a_0\, dt = \underline{\qquad\qquad}$

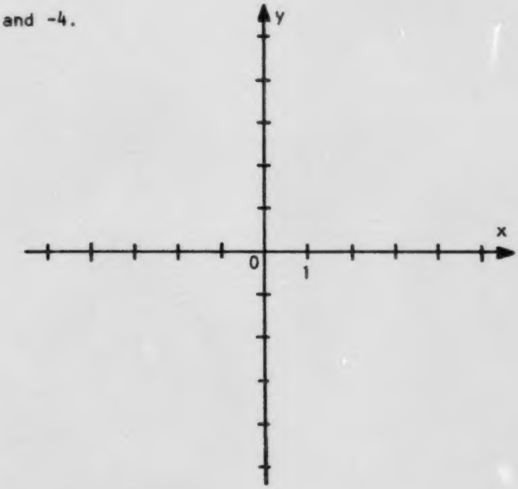
37. $\int v_0\, dt = \underline{\qquad\qquad}$

38. $\int \underline{\qquad\qquad} dy = y_0 y + C$

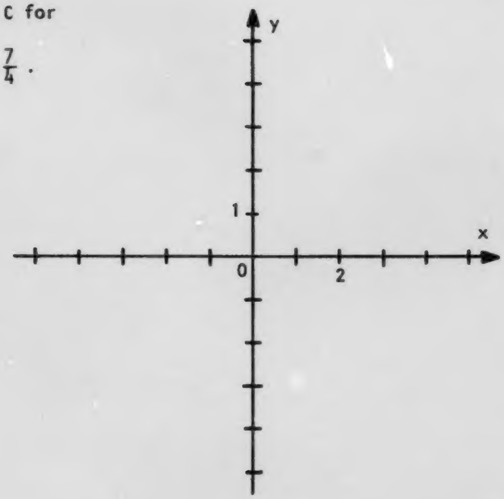
39. $\int \underline{\qquad\qquad} dt = s_0 t + C$

In Exercises 40-43, use the coordinate axes provided to graph the three curves selected from each family.

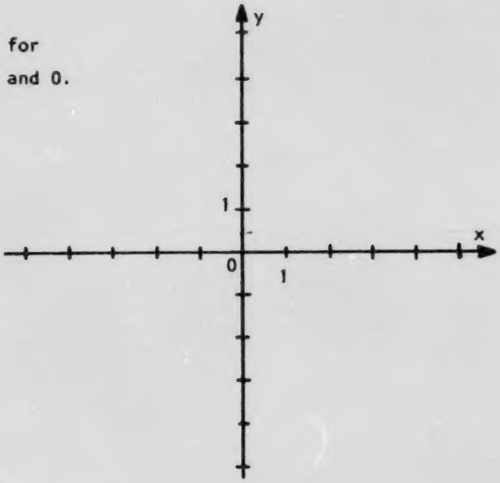
40. Graph $y = -2x + C$ for $C = 0$, 3, and -4.



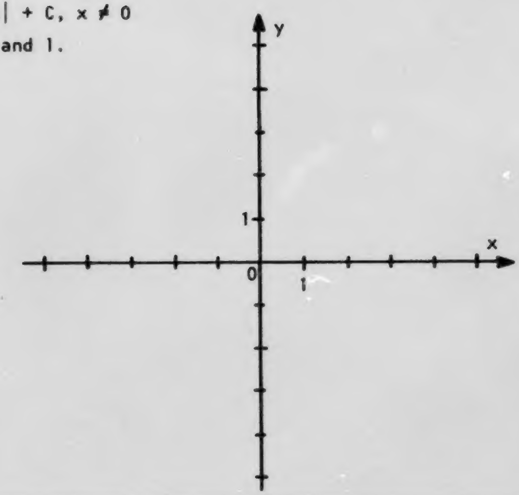
41. Graph $y = \frac{x^2}{4} + C$ for $C = 0$, -1, and $\frac{7}{4}$.



42. Graph $y = e^x + C$ for $C = -2.718$, -1, and 0.



43. Graph $y = \ln|x| + C$, $x \neq 0$ for $C = 0$, -1, and 1.



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3. INTEGRALS OF LINEAR COMBINATIONS OF FUNCTIONS

The rules about differentiating sums of functions and multiples of functions lead to the following two integration formulas:

(6) SUM RULE $\int [f(x) + g(x)]\, dx + \int g(x)\, dx$

(7) SCALAR MULTIPLE RULE $\int k\, f(x)\, dx = k \int f(x)\, dx$
(k any constant).

The fact that the integral of the negative of a function is the negative of its integral is an immediate consequence of (7). We just take $k = -1$ to get

$$\int -f(x)\, dx = \int -1 \cdot f(x)\, dx = -1 \cdot \int f(x)\, dx = -\int f(x)\, dx.$$

EXAMPLE 3.

$$\int -x^2\, dx = -\int x^2\, dx = -\frac{x^3}{3} + C$$

The sum and scalar multiple rules are often used together, as in the next example.

EXAMPLE 4.

$$\int (y^4 - \frac{1}{y} + 9)dy = \int y^4 dy - \int \frac{1}{y}\, dy + \int 9dy \quad \text{SUM RULE}$$

$$= \frac{y^5}{5} - \ln |y| + 9y + C \quad \text{SCALAR MULTIPLE RULE}$$

In general, the sum and scalar multiple rules allow us to break problems into parts we know how to solve (we hope).

When we integrate a sum or difference of functions, one constant of integration is enough to generate the whole family of possible solutions.

EXAMPLE 5.

$$\int (3t^2 + 12e^t)dt = \int 3t^2 dt + 12\int e^t\, dt$$

$$= t^3 + 12e^t + C.$$

We do not need to write the answer as $t^3 + C_1 + 12e^t + 12C_2$. The formulas $t^3 + 12e^t + C$ and $t^3 + C_1 + 12e^t + 12C_2$ may generate different functions for different values of the C's but the family of functions generated by either formula is the same as the family generated by the other. We are therefore free to use the simpler formula, which is what we do.

EXERCISES

Complete the equations in Exercises 44 - 55.

44. $\int(x+1)dx = \frac{x^2}{2} + \_\_\_\_\_$

45. $\int(4-x)dx = \_\_\_\_\_$

46. $\int \_\_\_\_\_ dx = \frac{x^2}{2} - x + C$

47. $\int \_\_\_\_\_ dx = -\ln|x| + C$

48. $\int -e^x dx = \_\_\_\_\_$

49. $\int -t^2 dt = -\frac{t^3}{3} + \_\_\_\_\_$

50. $\int(mx+b)dx = \_\_\_\_\_$

51. $\int(3y^2 - 5y)dy = \_\_\_\_\_$

52. $\int \_\_\_\_\_ ds = s^2 - s^3 + C$

53. $\int(\frac{1}{4}x - \frac{1}{3}e^x)dx = \_\_\_\_\_$

54. $\int(\frac{3}{z} - \frac{z}{3} + 4e^z)dz = \_\_\_\_\_$

55. $\int \_\_\_\_\_ dx = x^3 - x^2 + 7x + C$

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4. FRACTIONAL EXPONENTS AND NEGATIVE EXPONENTS

In this section we return to formula (3') of Section 2, which we now call the *power formula.*

(8) POWER FORMULA $\qquad \int x^r dx = \frac{x^{r+1}}{r+1}, \; r \neq -1$

We do this to point out that the exponent r in the formula does not have to be positive. It also does not have to be an integer.

EXAMPLE 6.

$$\int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

USE THE POWER FORMULA WITH $r = \frac{1}{2}$.

$$= \frac{2}{3}x^{\frac{3}{2}} + C$$

SIMPLIFY

EXAMPLE 7.

$$\int \frac{1}{x^4}dx = \int x^{-4}dx$$

WRITE THE INTEGRAND WITH A NEGATIVE EXPONENT. USE THE POWER FORMULA WITH $r = -4$.

$= \frac{x^{-3}}{-3} + C$	INTEGRATE
$= -\frac{x^{-3}}{3} + C$	POSITIVE DENOMINATOR
$= -\frac{1}{3x^3} + C$	POSITIVE EXPONENT

EXAMPLE 8.

$\int -\frac{15}{x^2}\,dx = -15\int \frac{1}{x^2}\,dx$	USE THE SCALAR MULTIPLE RULE TO SIMPLIFY THE INTEGRAND.
$= -15\int x^{-2}\,dx$	WRITE THE INTEGRAND IN EXPONENTIAL FORM. TAKE r = -2 IN THE POWER FORMULA.
$= -15\cdot\frac{x^{-1}}{-1} + C$	INTEGRATE
$= 15x^{-1} + C$	SIMPLIFY
$= \frac{15}{x} + C$	POSITIVE EXPONENT

EXERCISES

Complete the equations in Exercises 56-80.

56. $\int t^{\frac{1}{2}}\,dt =$ ______

57. $\int$ ______ $dx = 2x^{\frac{1}{2}} + C$

58. $\int \sqrt{x}\,dx =$ ______

59. $\int 5\sqrt{x}\,dx =$ ______

60. $\int -12\sqrt{s}\,ds =$ ______

61. $\int -2\sqrt{y}\,dy =$ ______

62. $\int \frac{1}{y^4}\,dy =$ ______

63. $\int \frac{4}{x^5}\,dx =$ ______

64. $\int$ ______ $dx = \frac{1}{x^2} + C$

65. $\int$ ______ $dx = \frac{1}{x} + C$

66. $\int$ ______ $dt = \frac{5}{t^3} + C$

67. $\int$ ______ $ds = \frac{-4}{s^2} + C$

68. $\int \frac{8}{y^3}\,dy =$ ______

69. $\int \frac{dv}{v^2} =$ ______

70. $\int \frac{1}{x}\,dx =$ ______

71. $\int \frac{-2}{x}\,dx =$ ______

72. $\int$ ______ $dx = x + \ln|x| + C$

73. $\int$ ______ $dv = v + \frac{1}{v} + C$

74. $\int (x^2 + \frac{1}{x^2})\,dx =$ ______

75. $\int (x + \frac{1}{x})\,dx =$ ______

76. $\int \_\_\_\_\_ dz = e^z + \ln|z| + C$ 77. $\int (e^x - \frac{1}{x^2})dx = \_\_\_\_\_$

78. $\int \frac{200}{z}\, dz = \_\_\_\_\_$ 79. $\int \frac{-602}{x^3}\, dx = \_\_\_\_\_$

80. $\int \frac{3}{2} \sqrt{y}\, dy = \_\_\_\_\_$

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5. DIFFERENTIATE TO CHECK

To check an indefinite integration there are two steps to follow:

1. Make sure a constant of integration is there.
2. Differentiate, to see if you get back the function you integrated.

EXAMPLE 9.

The equation

$$\int (2x^2 - 5x^4)dx = \frac{2}{3}x^3 - x^5 + C$$

is correct because

(1) C is there, and

(2) $\frac{d}{dx}(\frac{2}{3}x^3 - x^5 + C) = 3 \cdot \frac{2}{3}x^2 - 5x^4 + 0$

$= 2x^2 - 5x^4$.

EXERCISES

True, or false? Differentiate to find out.

81. $\int x^3 dx = \frac{x^4}{4} + C$ 82. $\int x^2 dx = x^3 + C$

83. $\int t^{\frac{1}{2}} dt = \frac{2}{3} t^{\frac{3}{2}} + C$ 84. $\int \sqrt{y}\, dy = y^{\frac{3}{2}} + C$

85. $\int (e^s - \frac{1}{s})\, ds = e^s - 1 + C$ 86. $\int \frac{1}{x^2}\, dx = -\frac{1}{x} + C$

Find the missing integrands in Exercises 87-98.

87. $\int \_\_\_\_\_ dx = x^5 + C$ 88. $\int \_\_\_\_\_ dt = 16t^2 + C$

89. $\int \_\_\_\_\_ dy = \frac{1}{y} + C$ 90. $\int \_\_\_\_\_ = dt = 16t^2 + v_0 t + s_0$

(Here, v_0 and s_0 are constants.)

91. $\int \_\_\_\_\_ dy = \ln |y| + C$

92. $\int \_\_\_\_\_ dt = 32t + v_0$
(v_0 is a constant.)

93. $\int \_\_\_\_\_ dx = mx + b$
(m and b are constants.)

94. $\int \_\_\_\_\_ dx = x^{\frac{1}{2}} - x + C$

95. $\int \_\_\_\_\_ dT = \frac{600}{T} + T_0$
(T_0 is a constant.)

96. $\int \_\_\_\_\_ dx = 23\, e^{x} + C$

97. $\int \_\_\_\_\_ dx = -120x^2 - 70x + C$

98. $\int \_\_\_\_\_ dv = \frac{2}{3} v^{\frac{1}{2}} + C$

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6. NUMERICAL CONDITIONS THAT DETERMINE A CONSTANT OF INTEGRATION

Every function f whose derivative is given by the formula

$$f'(x) = 10x$$

is a member of the family of functions

$$\int 10x\,dx = 5x^2 + C.$$

But to determine just which one f is, we need more information. The information can be supplied in various ways. For instance, we might know the value of f at a particular value of x, as in the following example.

EXAMPLE 10. Find f if $f'(x) = 10x$ and $f(1) = 3$.

SOLUTION

1. $f(x) = 5x^2 + C$ for some C. BECAUSE $f'(x) = 10x$.

2. $5(1)^2 + C = 3$ BECAUSE $f(1) = 3$.
$5 + C = 3$
$C = -2$

Conditions like $f(1) = 3$ that select a *particular solution* from a family given by an indefinite integral are called *initial conditions*.

EXERCISES

Copy and complete the table below. The first row, already complete, shows the following information:

1. The derivative of a function
2. The indefinite integral of this derivative
3. An initial condition
4. The value of C determined by the given initial condition
5. The function determined by the derivative together with the initial condition.

$\boxed{2x}$ $\boxed{x^2 + C}$ $\boxed{f(-5) = 29}$ $\boxed{4}$ $\boxed{f(x) = x^2 + 4}$

	DERIVATIVE	INDEFINITE INTEGRAL	INITIAL CONDITION	VALUE OF C	PARTICULAR SOLUTION
	$f'(x)$	$\int f'(x)dx$	$f(x_0)$	C	$f(x)$
	$2x$	$x^2 + C$	$f(-5) = 29$	4	$f(x) = x^2 + 4$
99.		$\ln \lvert x\rvert + C$	$f(e) = -3$		
100.	$\frac{1}{x}$		$f(1) = 2$		
101.	$-x$		$f(1) = 0$		
102.	$x^2 + 6$		$f(1) = 10$		
103.	32		$f(0) = 0$		
104.	v_0		$f(0) = 5$		
105.	$32x + v_0$		$f(0) = 0$		

106. Find the potential energy $U(x)$ of an object as a function of its position x, when the magnitude $F(x)$ of the force acting on the object is given by

$$F(x) = -kx.$$

Assume that $U(0) = 0$ and that $U(x) = \int F(x)\, dx$.

7. GRAPHICAL CONDITIONS THAT DETERMINE A CONSTANT OF INTEGRATION

The graphs of the functions

$$f(x) = \int (4 - 3x^2)dx = 4x - x^3 + C$$

make a family of non-overlapping curves in the plane. There is one curve for each value of C.

Choosing a function from the family amounts to choosing one of these curves.

One way to pick out a curve is to name a point on it. We might say, for example, "Take the curve that passes through the point (0,3)." This says that, of all the curves $y = f(x)$, we want the one that satisfies the initial condition $f(0) = 3$.

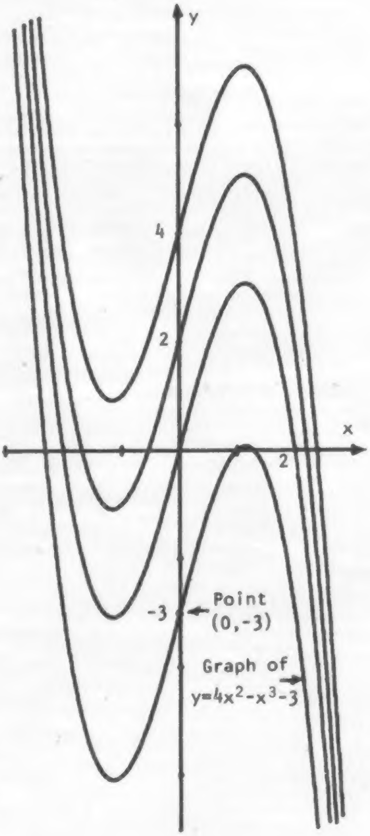


Figure 2. The graphs of $y = 4x - x^3 + c$, for $C = -3, 0, 2, 4$.

EXAMPLE 11. Find f if

1. $f'(x) = 4 - 3x^2$.
2. The graph of f passes through (1,5).

SOLUTION

1. The values of f are given by the formula

$$f(x) = \int (4 - 3x^2)dx = 4x - x^3 + C.$$

2. The graph of f has the equation

$$y = 4x - x^3 + C.$$

3. Because (1,5) lies on the graph,

$$\begin{aligned} 4(1) - (1)^3 + C &= 5 \\ 3 + C &= 5 \\ C &= 2. \end{aligned}$$

4. Thus, $f(x) = 4x - x^3 + 2$.

EXERCISES

In Exercises 107-112, find the value of C that makes the curve $y = 4x - x^3 + C$ pass through the given point.

107. (0,0)
108. (2,0)
109. (-2,0)
110. (0,4)
111. (2,7)
112. (3,1)

Copy and complete the table on page 17.

	FORMULA FOR $f'(x)$	$\int f'(x)dx$	A POINT ON THE GRAPH OF f	FORMULA FOR $f(x)$
	$2x$	$x^2 + C$	(5,20)	$x^2 - 5$
113.	5		(-2,1)	
114.	$8x$		$(0,\sqrt{2})$	
115.	$-4x + 3$		(-1,1)	
116.	$9.8x$		(1,3)	
117.	$e^x - 2$		(0,7)	
118.	$\frac{6}{x}$		(1,4)	
119.	$-\frac{3}{x^2}$		(2,0)	
120.	$\sqrt{x}$		(3,0)	

In Exercises 121-124, graph the function whose derivative is given and that satisfies the given initial condition.

121. $\frac{ds}{dt} = 32t + 10$

$s(0) = -4$

122. $\frac{dv}{dt} = 9.8$

$v(0) = 0$

123. $\frac{dy}{dt} = 5e^t$

$y(0) = 7$

124. $\frac{ds}{dt} = 9.8t$

$s(0) = 0$

8. MODELING: INITIAL CONDITIONS FROM PLAUSIBLE ASSUMPTIONS

When initial conditions are not stated explicitly, they can sometimes be inferred from other information or based on plausible assumptions.

EXAMPLE 12. To sample the upper atmosphere a rocket is fired straight up from the ground. The rocket engine accelerates the rocket at 4m/sec^2, and has enough fuel to burn for 2 minutes.

1) How high is the rocket 1 minute after launch?
2) How fast is it climbing then?
3) How high will the rocket be when the engine stops?
4) How fast will it be climbing then?

ANALYSIS

The questions on the list are not as formidable as they might seem at first glance because we can answer them all by finding formulas that describe the rocket's height and speed as functions of time.

To begin, let $s(t)$ denote the rocket's height in meters as a function of time measured in seconds. The choice of meters and seconds is a natural one to make, because the rocket's acceleration is given in those terms. The use of the letter s is traditional.

Then $s'(t)$ gives the rocket's velocity and $s''(t)$ its acceleration, so that while the engine is on

$$s''(t) = 4 \text{ m/sec}^2.$$

If we measure time with $t = 0$ at the time of ignition, and assume that the engine gives full thrust from the very start, then,

$$s''(t) = 4 \qquad 0 \leq t \leq 120$$

and

$$\begin{aligned} s'(t) &= \int s''(t)dt \\ &= \int 4dt \\ &= 4t + C, \ 0 \leq t \leq 120, \end{aligned}$$

meters per second being understood. Since $C = s'(0)$ is the *initial velocity* of the rocket, we usually write v_0 in place of C, as in the next equation:

$$s'(t) = 4t + v_0, \; 0 \leq t \leq 120.$$

If we assume that the rocket is fired from rest, then

$$v_0 = 0 \qquad \text{and} \qquad s'(t) = 4t \text{ when } 0 \leq t \leq 120.$$

To find $s(t)$ for the two-minute interval the engine is on we integrate again. This gives

$$\begin{aligned} s(t) &= \int s'(t)\, dt \\ &= \int 4t\, dt \\ &= 2t^2 + s_0 \text{ meters}, \; 0 \leq t \leq 120. \end{aligned}$$

Notice that s_0 is $s(0)$, the so-called *initial distance*. To assign a value to it we assume that distance is measured up from the launching pad with $s(0) = 0$. Accordingly,

$$s(t) = 2t^2, \qquad 0 \leq t \leq 120,$$

and the rocket's motion is completely described for the first two minutes of flight.

We will take up what happens to the rocket after burnout when we get to the next exercises. For the moment, let us look again at the decisions and assumptions we have made, and how they enabled us to calculate the rocket's height as a function of time.

We first decided on a notation $s(t)$ for the rocket's height as a function of time. In terms of this notation we wrote $s'(t)$ for the velocity and $s''(t)$ for the acceleration. Then we made assumptions about how the rocket worked and how it was launched, and decisions about how time and distance were to be measured. These translated into numerical data about s(t) and its derivatives, and lead to a description of the motion during the "burn" *period*.

ASSUMPTIONS DECISIONS	GENERATED DATA	CONCLUSIONS ABOUT THE MOTION DURING THE BURN PERIOD
Time is measured in seconds and s(t) in meters. The engine is on for $0 \leq t \leq 120$. The engine gives full thrust while on.	$s''(t) = 4$ $0 \leq t \leq 120$	$s'(t) = 4t + v_0$ $0 \leq t \leq 120$
The rocket is fired from rest.	$v_0 = 0$	$s'(t) = 4t + 0 = 4t$ $s(t) = 2t^2 + s_0$ $0 \leq t \leq 120$
Distance is measured up from the launching pad.	$s_0 = 0$	$s(t) = 2t^2 + 0 = 2t^2$ $0 \leq t \leq 120$

EXERCISES

We now use the equations for s(t) and s'(t) to answer the questions with which we began this section. The table that follows shows how the first two questions can be rephrased in terms of s(t) and s'(t), and then answered with simple calculations. Do the same for the remaining questions.

QUESTION	REPHRASED IN TERMS OF THE MODEL	ANSWERED
How high is the rocket 1 minute after launch?	$s(60) = ?$	$s(60) = 2(60)^2$ m $= 7200$ m $= 7.2$ km
How fast is it climbing one minute after launch? a) in m/sec b) in km/h	$s'(60) = ?$	$s'(60) = 4(60)$ m/sec $= 240$ m/sec $= 864$ km/h
125. How high will the rocket be when the engine stops? a) in meters b) in kilometers		
126. How fast will it be climbing when the engine stops?		
127. When will the rocket be 20 km above the launch site?	For what t is $s(t) = 20{,}000$ m?	
128. How long does it take the rocket to reach a velocity of 100m/sec?		
129. How long did it take the rocket to rise the first 50 m? Can a good runner run 50 m that fast?		
130. How long did it take the rocket to travel the next 50 m?		

As you gain experience, you will not write as much as we did when we discussed the rocket problem. You might not need to write more than

$$\begin{aligned} s''(t) &= 4 && \text{m/sec}^2 \\ s'(t) &= 4t + v_0 = 4t && \text{m/sec} \\ s\,(t) &= 2t^2 + s_0 = 2t^2 && \text{m} \end{aligned}$$

Remember to write down the units, though, if only for later reference.

It was convenient to have had the initial velocity v_0 and the initial distance s_0 both equal to zero. This allowed us to describe the velocity and distance traveled by the rocket during the burn period simply, by the equations $s'(t) = 4t$ and $s(t) = 2t^2$. It will not always be possible to make v_0 and s_0 both zero in describing a motion, however, nor will making them zero always be desirable. Exercise 131 below is a case in point.

EXERCISES

There is more to be learned about the flight of the rocket. The rocket coasts upwards for a while after the engine shuts off. For how long? And how high?

To answer these questions we need a mathematical model that is different from the one we have been using. The reason for this is that when the engine shuts down the force acting on the rocket changes. The acceleration of the rocket is no longer 4 m/sec^2 upwards provided by the engine, but rather 9.8 m/sec^2 downwards, provided by the earth's gravitational attraction.

If we continue to measure distance as before, but reset our clock to start with $t = 0$ again, the equation for the acceleration becomes

$$s''(t) = -9.8 \text{ m/sec}^2 \qquad 0 \leq t$$

so that

$$\begin{aligned} s'(t) &= 9.8t + v_0 \text{ m/sec} && 0 \leq t \\ s(t) &= -4.9t^2 + v_0\,t + s_0 \text{ m} && 0 \leq t \end{aligned}$$

131. a) What numerical values should v_0 and s_0 have?

b) Rewrite the equations just given for $s'(t)$ and $s(t)$ using the initial values from Part a.

Now complete the following table.

QUESTION	REPHRASED IN TERMS OF THE MODEL	ANSWERED
132. How long does the rocket coast upwards after burnout?		
133. How high does the rocket go?		
134. When do the equations of motion predict the rocket will crash?		
135. What is the rocket's predicted speed just before it crashes?		
136. Would you expect a real rocket to behave as predicted in Exercises 134 and 135? Explain.		

9. REPEATED INTEGRATION

As you saw in the preceding section, when we integrate more than once to solve a problem we need a corresponding number of initial conditions to determine the constants of integration. Here are two more examples.

EXAMPLE 13. Find $f(x)$ if $f''(x) = 12x - 14$, $f'(0) = 5$ and $f(0) = -3$.

SOLUTION The initial conditions can be used one at a time as the integration proceeds, or at the end of the integration.

METHOD #1

1. Integrate $f''(x) = 12x - 14$
   to get $f'(x) = 6x^2 - 14x + C_1$.

2. Use the condition $f'(0) = 5$ to find C_1.

$$6(0) - 14(0) + C_1 = 5$$
$$C_1 = 5$$

3. Integrate $f'(x) = 6x^2 - 14x + 5$
   to get $f(x) = 2x^3 - 7x^2 + 5x + C_2$.

4. Use the condition $f(0) = -3$ to find C_2.

$$2(0)^3 - 7(0)^2 + 5(0) + C_2 = -3$$
$$C_2 = -3.$$

5. Write the completed formula for $f(x)$.

$$f(x) = 2x^3 - 7x^2 + 5x - 3.$$

METHOD #2

1. Integrate twice.

$$f''(x) = 12x - 14$$
$$f'(x) = 6x^2 - 14x + C$$
$$f(x) = 2x^3 - 7x^2 + C_1x + C_2.$$

2. Substitute $x = 0$ in the last two equations to find C_1 and C_2.

$$C_1 = f'(0) = 5$$
$$C_2 = f(0) = -3$$

3. Write the formula for $f(x)$.

$$f(x) = 2x^3 - 7x^2 + 5x - 3$$

EXAMPLE 14. Find $f(x)$ if $f''(x) = 4x - 2$ and if the graph of f passes through the point (1,0) with slope 3.

SOLUTION

1. Integrate $f''(x)$ twice.

$$f''(x) = 4x - 2$$
$$f'(x) = 2x^2 - 2x + C_1$$
$$f(x) = \frac{2}{3}x^3 - x^2 + C_1x + C_2.$$

2. Determine C_1 from the fact that the slope of the graph is 3 when $x = 1$.

$$f'(1) = 3$$
$$2(1)^2 - 2(1) + C_1 = 3$$
$$C_1 = 3$$

3. Substitute this value of C_1 in the expression for $f(x)$.

$$f(x) = \frac{2}{3}x^3 - x^2 + 3x + C_2.$$

4. Determine C_2 from the fact that $f(1) = 0$.

$$f(1) = 0$$
$$\frac{2}{3}(1)^3 - (1)^2 + 3(1) + C_2 = 0$$
$$\frac{8}{3} + C_2 = 0$$
$$C_2 = -\frac{8}{3}.$$

5. Write $f(x)$.

$$f(x) = \frac{2}{3}x^3 - x^2 + 3x - \frac{8}{3}.$$

EXERCISES

Find the function determined by each set of conditions.

137. $f''(x) = 2-6x$; $f'(0) = 4$, $f(0) = 1$.

138. $g''(t) = 30t$; $g'(1) = 0$, $g(1) = 10$.

139. $h''(x) = e^x$; $h'(0) = h(0) = 1$.

140. $k''(y) = \sqrt{y}$; $k'(9) = 10$, $k(0) = 8$.

141. $p'''(t) = 6$; $p''(0) = -8$, $p'(0) = 0$, $p(0) = 5$.

142. $f''(t) = 1 - 6t$, and the graph of f passes through the point $(2,0)$ with slope 0.

143. $r''(x) = \frac{3s}{8}$, and the graph of r passes through the point $(4,4)$ with slope 3.

144. $g''(x) = e^x$, and the graph of g passes through the origin with slope 2.

145. $h''(x) = -\frac{1}{x^2}$, $x > 0$, and the graph of h passes through $(1,2)$ with slope 0.

---

10. ANSWERS TO EXERCISES

Section 1

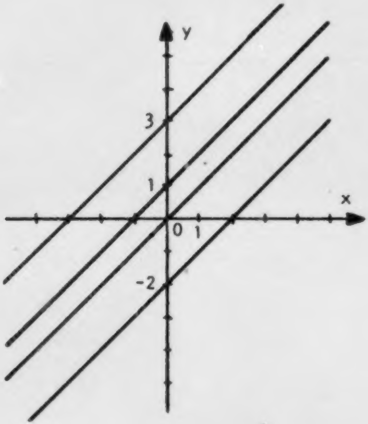
1. $x = 0, 1$

2. b) $\frac{d}{dx}(2x^3 - 3x^2) = 6x^2 - 6x$

   is not a constant on any interval.

3. Some examples are: $5x^4$ and $4x^5$; $\sin x$ and $\cos x$.

4.



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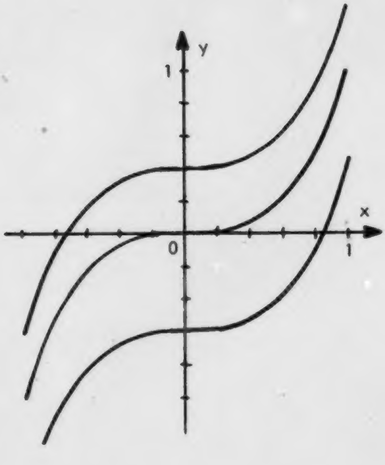


Section 2

6. $4x + C$

7. $-25x + C$

8. 17

9. -3

10. $\frac{x^3}{3} + C$

11. $\frac{x^6}{6} + C$

12. C

13. $\ln |x| + C$

14. x^7

15. $\frac{x^4}{12} + C$

16. $x^{103} + C$

17. x^{16}

18. e^x

19. $4x^3$

20. x^3

21. -103.5

22. $kx + C$

23. $ax + C$

24. m

25. -b

26. $32t + C$

27. a

28. $\frac{p^3}{3} + C$

29. $3B^2$

30. $s^5 + C$

31. $\frac{v^2}{2} + C$

32. $e^y + C$

33. $\frac{1}{R}$

34. $t_0t + C$

35. $32t_0$

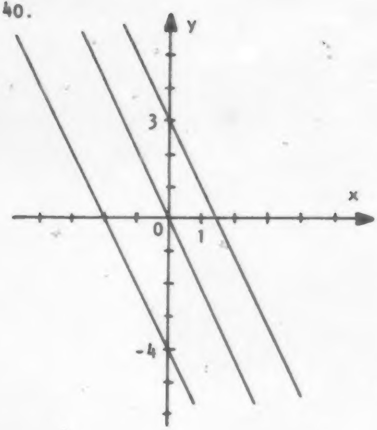
36. $a_0t + C$

37. $v_0t + C$

38. y_0

39. s_0

40.



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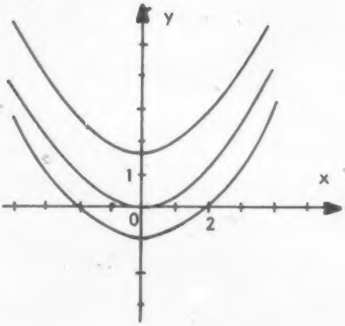

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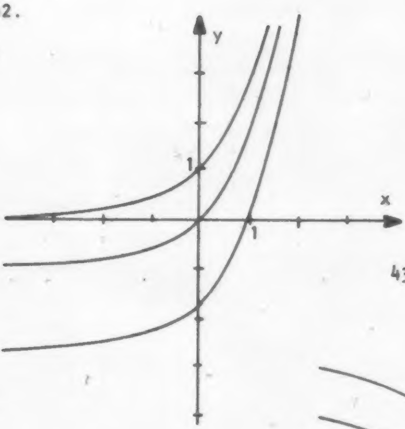

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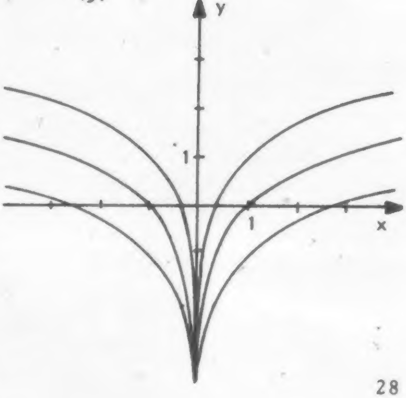


Section 3

44. $x + C$

45. $4x - \frac{x^2}{2} + C$

46. $x - 1$

47. $-\frac{1}{x}$

48. $-e^x + C$

49. C

50. $m\frac{x^2}{2} + bx + C$

51. $y^3 - \frac{5}{2}y^2 + C$

52. $2s - 3s^2$

53. $\frac{x^2}{8} - \frac{1}{3}e^x + C$

54. $3\ \ln|z| - \frac{z^2}{6} + 4e^z + C$

55. $3x^2 - 2x + 7$

Section 4

56. $\frac{2}{3}t^{\frac{3}{2}} + C$

57. $x^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{x}}$

58. $\frac{2}{3}x^{\frac{3}{2}} + C$

59. $\frac{10}{3}x^{\frac{3}{2}} + C$

60. $-8s^{\frac{3}{2}} + C$

61. $-\frac{4}{3}y^{\frac{3}{2}} + C$

62. $-\frac{1}{3y^3} + C$

63. $-\frac{1}{x^4} + C$

64. $-\frac{2}{x^3}$

65. $-\frac{1}{x^2}$

66. $-\frac{15}{t^4}$

67. $\frac{8}{s^3}$

68. $-\frac{4}{y^2} + C$

69. $-\frac{1}{v} + C$

70. $\ln|x| + C$

71. $-2\ \ln|x| + C$

72. $1 + \frac{1}{x}$

73. $1 - \frac{1}{v^2}$

74. $\frac{x^3}{3} - \frac{1}{x} + C$

75. $\frac{x^2}{2} + \ln|x| + C$

76. $e^z + \frac{1}{z}$

77. $e^x + \frac{1}{x} + C$

78. $200\ \ln|z| + C$

79. $\frac{301}{x^2} + C$

80. $y^{\frac{3}{2}} + C$

Section 5

81. True
82. False
83. True
84. False
85. False
86. True
87. $5x^4$
88. 32t
89. $-\frac{1}{y^2}$
90. $32t + v_0$
91. $\frac{1}{y}$
92. 32
93. m
94. $\frac{1}{2\sqrt{x}} - 1$
95. $-\frac{600}{T^2}$
96. $23\ e^x$
97. $-240x - 70$
98. $\frac{1}{3\sqrt{v}}$

	DERIVATIVE	INDEFINITE INTEGRAL	INITIAL CONDITION	VALUE OF C	PARTICULAR SOLUTION
	$f'(x)$	$\int f'(x)dx$	$f(x_0)$	C	$f(x)$
	$2x$	$x^2 + C$	$f(-5) = 29$	4	$f(x) = x^2 + 4$
99.	$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$f(e) = -3$	-4	$f(x) = \ln\lvert x\rvert - 4$
100.	$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$f(1) = 2$	2	$f(x) = \ln\lvert x\rvert + 4$
101.	$-x$	$-\frac{x^2}{2} + C$	$f(1) = 0$	$\frac{1}{2}$	$f(x) = -\frac{x^2}{2} + \frac{1}{2}$
102.	$x^2 + 6$	$\frac{x^3}{3} + 6x + C$	$f(1) = 10$	$\frac{11}{3}$	$f(x) = \frac{x^3}{3} + 6x + \frac{11}{3}$
103.	32	$32x + C$	$f(0) = 0$	0	$f(x) = 32x$

104.	v_0	$v_0x + C$	$f(0) = 0$	0	$f(x) = v_0x + 5$
105.	$32x + v_0$	$16x^2 + v_0x + C$	$f(0) = 0$	0	$f(x) = 16x^2 + v_0x$
106.	$U(x) = -\frac{kx^2}{2}$				

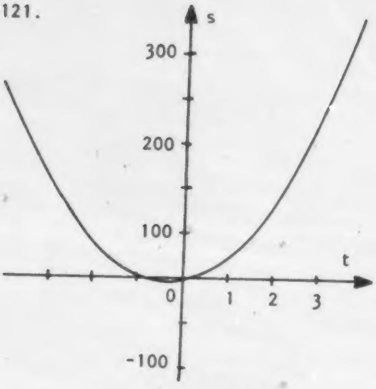
Section 7

107. 0 108. 0 109. 0

110. 4 111. 7 112. 16

	FORMULA FOR $f'(x)$	$\int f'(x)dx$	A POINT ON THE GRAPH OF F	FORMULA FOR $f(x)$
	$2x$	$x^2 + C$	(5, 20)	$x^2 - 5$
113.	5	$5x + C$	(-2, 1)	$5x + 11$
114.	$8x$	$4x^2 + C$	$(0, \sqrt{2})$	$4x^2 + \sqrt{2}$
115.	$-4x + 3$	$-2x^2 + 3x + C$	(-1, 1)	$-2x^2 + 3x + 6$
116.	$9.8x$	$4.9x^2 + C$	(1, 3)	$4.9x^2 - 1.9$
117.	$e^x - 2$	$e^x - 2x + C$	(0, 7)	$e^x - 2x + 6$
118.	$\frac{6}{x}$	$6 \ln\|x\| + C$	(1, 4)	$6 \ln\|x\| + 4$
119.	$-\frac{3}{x^2}$	$\frac{3}{x} + C$	(2, 0)	$\frac{3}{x} - \frac{3}{2}$
120.	$\sqrt{x}$	$\frac{2}{3} x^{\frac{3}{2}} + C$	(3, 0)	$\frac{2}{3} x^{\frac{3}{2}} - 2\sqrt{3}$

121.



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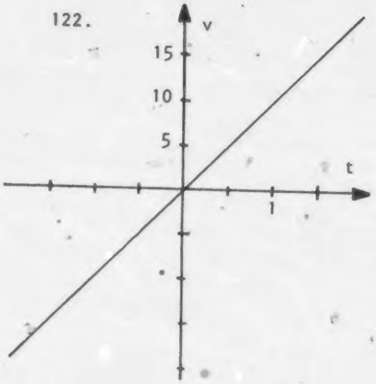

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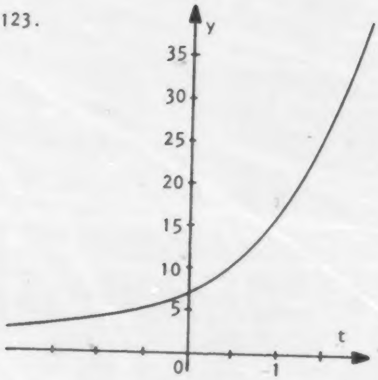

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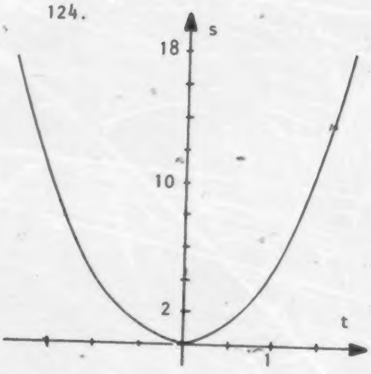

	QUESTION	REPHRASED IN TERMS OF THE MODEL	ANSWERED
	How high is the rocket 1 minute after launch?	$s(60) = ?$	$s(60) = 2(60)^2$ m $= 7200$ m $= 7.2$ km
	How fast is it climbing one minute after launch? a) in meters b) in kilometers	$s'(60) = ?$	$s'(60) = 4(60)$ m/sec a) $= 240$ m/sec b) $= 864$ km/h
125.	How high will the rocket be when the engine stops? a) in meters b) in kilometers	$s(120) = ?$	$s(120) = 2\ (120)^2$ m a) $= 28{,}800$ m b) $= 28.8$ km
126.	How fast will it be climbing when the engine stops? (The speed of sound in air at sea level is about 335 m/sec. Speeds of about 500 m/sec are typical for oxygen molecules at room temperature.	$s'(120) = ?$	$s'(120) = 4(120)$ m/sec a) $= 480$ m/sec b) $= 1728$ km/h
127.	When will the rocket be 20 km above the launch site?	For what t is $s(t) = 20{,}000$ m?	$2t^2 = 20{,}000$ $t^2 = 10{,}000$ $t = 100$ sec
128.	How long does it take the rocket to reach a velocity of 100 m/sec?	For what t is $s'(t) = 100$ m/sec?	$4t = 100$ $t = 25$ sec
129.	How long did it take the rocket to rise the first 50 m? Can a good runner run 50 m that fast?	For what t is $s(t) = 50$ m?	$2t^2 = 50$ $t^2 = 25$ $t = 5$ sec The current world record for 100 m is 9.9 seconds.

	QUESTION	REPHRASED IN TERMS OF THE MODEL	ANSWERED
130.	How long did it take the rocket to travel the next 50 m?	Find the t for which s(t) = 100 m. Then subtract 5 sec.	a) $2t^2 = 100$ $t^2 = 50$ $t = 5\sqrt{2}$ sec b) $5\sqrt{2} - 5 = 5(\sqrt{2}-1)$ $\approx 5(0.414)$ ≈ 2.07 sec

131. a) $v_0 = 480$ m/sec, and $s_0 = 28,800$ m (See Exercises 126 and 125).

b) $s'(t) = -9.8t + 480$ m/sec

$s(t) = -4.9t^2 + 480t + 28,800$ m

	QUESTION	REPHRASED IN TERMS OF THE MODEL	ANSWERED
132.	How long does the rocket coast upwards after burnout?	For what t is $s'(t) = 0$?	$0 = -9.8t + 480$ $t = 480/9.8$ sec ≈ 49 sec
133.	How high does the rocket go?	$s(49) = ?$	$s(49) \approx 40,555$ m
134.	When do the equations of motion predict the rocket will crash?	For what t is $s(t) = 0$?	$t \approx 140$ sec after burnout.
135.	What is the rocket's predicted speed just before it crashes?	$s'(140) \approx ?$	-892 m/sec, or 892 m/sec downwards ($\approx$3,211.2 km/h)

136. No. The predictions are sure to be underestimates. They neglect the air resistance that will slow the rocket's fall, assume the rocket does not fragment, and so forth.

Section 9

137. $f(x) = -x^3 + x^2 + 4X + 1$

138. $g(t) = 5t^3 - 15t + 20$

139. $h(x) = e^x$

140. $k(y) = \frac{4}{15} y^{\frac{5}{2}} - 8y + 8$

141. $p(t) = t^3 - 4t^2 + 5$

142. $f(t) = \frac{t^2}{2} - t^3 + 10t - 14$

143. $r(s) = \frac{s^3}{16}$

144. $g(x) = e^x + x - 1$

145. $h(x) = \ln|x| - x + 3$